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of Greek mathematics. The work presented here suggests a new way of examining old constructive mathematics and a new way for historians of mathematics and philologists to do their research.

The work presented here also has implications for education. It has already been argued that Euclidean construction problems provide an excellent method of teaching high school students constructive proofs of existence theorems [Av89]. The work presented here suggests that Euclidean constructive geometry can be used as an ideal medium for teaching many of the most modern concepts concerning the design and analysis of algorithms, to high school students. For easy problems the students can prove that Euclid's constructions are valid for all possible inputs. For more difficult problems they can search for constructions that require fewer steps. Finally, for real challenging problems they can search for constructions that require the fewest number of steps.

10. Acknowledgment

The author is grateful to Hossam ElGindy for translating Ibn al-Haytham's 11th century Arabic manuscript [Ha1041], to Mariza Komioti for translating Heiberg's definitive Greek edition [Ka78], to Kelly Lyons for bringing to his attention a copy of Peyrard's book at Queens University [Pe1819], to Diane Souvaine, Sue Whitesides and Chee Yap for discussions on this topic as well as to Arnon Avron for providing some useful references.

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tion presented also in Pedoe [Pe76].

Algorithm CO: [Compass Only version]

Input: Let A be the given point, and BC the given straight line. {Thus it is required to place at the point A (as an extremity) a straight line equal to the given straight line BC.} See Fig. 8.2.

Begin

Step 1: Draw a circle with center A and radius AB.

Step 2: Draw a circle with center B and radius BA. {the two circles intersect at D and E.

Step 3: Draw a circle with center D and radius DC.

Step 4: Draw a circle with center E and radius EC.

These two circles intersect at C and X where X is the desired reflection point of C across the imaginary line through DE and XA is the desired length.

End

In the spirit of Proclus we invite the reader to supply the proofs of correctness of the above two constructions.

9. Conclusions

We mention in closing that even the 20th century **Algorithm CO** pales by comparison with **Algorithm Euclid** from the point of view of robustness with respect to singularities. Consider for example the case where point A happens to lie at a location equidistant from B and C. **Algorithm Euclid** executes in this case as easily as in any other since everything is well defined. Without special flag-waving code however **Algorithm CO** could crash attempting to draw a circle with radius zero and then intersecting two circles one of which has radius zero.

One apparent difference between modern and classical computational geometry concerns the issue of lower bounds on the complexity of geometric problems. Although Lemoine [Le02] and others were concerned with defining primitive operations and counting the number of such operations involved in a construction they did not ever appear to have considered the question of determining the minimum number of operations required to solve a given problem under a specified model of computation. For example, if we define 1) drawing a line and 2) drawing a circle, as the primitive operations allowed under the straight edge and compass model of computation, **Algorithm Euclid** takes nine steps, **Algorithm MS** takes six steps whereas **Algorithm CO** takes only four steps. Its non robustness not withstanding, is **Algorithm CO** *optimal*? In other words is four a lower bound on this problem? Is **Algorithm Euclid** the optimal *robust* algorithm? It is not difficult to argue that at least three steps are required. We conjecture that in fact four are always necessary.

This research suggests that perhaps the chaotic situation described here with respect to Euclid's second proposition exists also for his other propositions involving cases and indeed for all *straight line.*

Algorithm MS: [Mirror Symmetry version]

Input: Let A be the given point, and BC the given straight line. {Thus it is required to place at the point A (as an extremity) a straight line equal to the given straight line BC.} See Fig. 8.1.

Begin

Step 1: Draw a circle with center A and radius AB.

Step 2: Draw a circle with center B and radius BA.

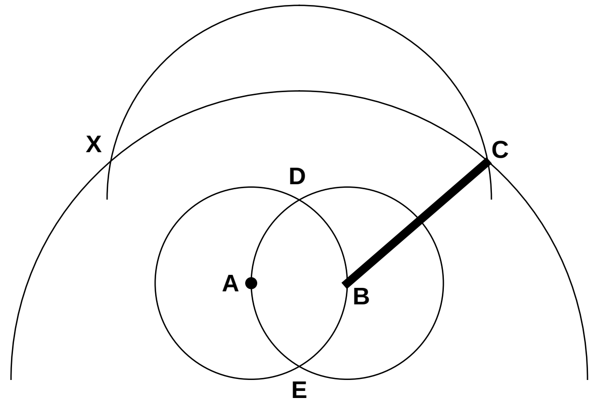

Fig. 8.2 Illustrating the construction with compasses only.

Step 3: Draw a line L through the intersection points D and E of the two circles produced in steps 1 and 2.

Step 4: Draw a circle with center C that intersects line L at points F and G.

Step 5: Draw a circle with center G and radius GC.

Step 6: Draw a circle with center F and radius FC.

These two circles intersect at C and H where H is the desired reflection point of C across L and HA the desired segment.

End

Recall that in 1672 Jorg Mohr and in 1797 the Italian geometer Lorenzo Mascheroni independently proved that any construction that can be carried out with a straight edge and a compass can be carried out with a compass alone. The reader may wonder how on earth we can draw a line segment of length BC with one extremity at A *without* using a straight edge. Strictly speaking we cannot and therefore in constructions with compasses alone we require only that in order to draw a line or line segment two points on the line or the two endpoints of the line segment be specified. Such a pair of points clearly specifies a line or line segment, as the case may be, in an unambiguous manner. Thus we are actually *simulating* a line or line segment by two points. In this sense it is more appropriate to state the Mohr-Mascheroni theorem as: *any construction that can be carried out with a straight edge and a compass can be* simulated *with a compass alone*. The above construction based on the principle of mirror symmetry uses both a straight edge and a compass. It is fitting to end this discussion by presenting a construction, also based on the mirror symmetry principle, that uses a compass only. We present the one described in [Ho70] which is the first construc-

of the text.

Accordingly, this French translation contains the following description of *Step 3* [refer to **Algorithm Euclid** and Fig. 4.1] *"Prolongeons DA suivant AE et BD suivant BF."* This literally means "Extend DA following AE and BD following BF." Note that the first half of this statement (extend DA and BD) is ambiguous and represents a step towards agreement with the phrasing of the texts of the 17th, 18th and 19th Century English texts. The complete statement is nevertheless saved by explicitly mentioning the A and B in AE and BD, respectively. One can easily envision these references to A and B being dropped during the next translation.

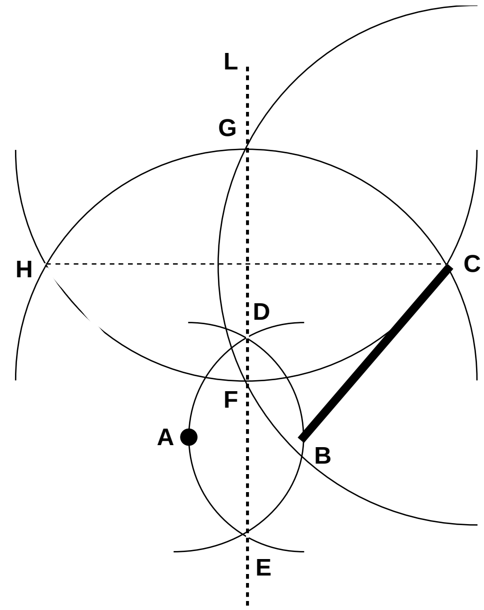

Fig. 8.1 Illustrating the mirror symmetry method for solving proposition 2.

A *literal* translation of the same Greek text yielded the following for *Step. 3: "Let lines AE, BF emerge outwards colinear to lines DA, DB."* Note that this is considerably more precise than the French version and is in agreement with the correct algorithms discussed earlier.

In conclusion, it is ironic that in Kayas' translation the desire to be true to the spirit of Euclid leads him on the road to betray Euclid. This suggests that although literal translations may fail miserably for poetry they may be essential for mathematics and computer science. The best remedy however is for the translator to possess an understanding of the deep structure of the proof.

8. Late 20th Century Algorithms

For the sake of comparison, contrast and completeness we offer in this section two alternate modern constructions that are fundamentally different from all those considered by Euclid, Heron, Proclus and the other Greek and subsequent commentators as well as the plethora of 19th and early 20th century text-book writers. These are based on the notion of *mirror symmetry.*

As before let A be the given point and let BC be the desired length to be transferred so that A is at an extremity. Without loss of generality let B be the chosen extremity and refer to Fig. 8.1. The idea is to first construct a line L that perpendicularly bisects the segment AB and subsequently perform a mirror symmetry transformation of the segment BC with line L as the axis of symmetry. Note that A and C may or may not be on the same side of L depending on the particular case of the input configuration at hand. In either case we reflect C about L to obtain our desired result. Note also that Euclid's construction does not necessarily yield a transferred length that is symmetrical about L.

Proposition 2: *To place at a given point* (*as an extremity*) *a straight line equal to a given* but the proofs are very often replaced by instructions for proofs or outlines of proofs."

Adelard of Bath writes *Step 3* as follows:

"*Protrahanturque linee DA et DB directe usque ad L et G.*"

His actual letters are different and are here substituted to match those of Fig. 4.1 for ease of discussion. As a minor aside, there is an error (probably typographic?) in this manuscript, i.e., L and G are actually reversed. More seriously, E and F are nonexistent, as are the references to producing the lines AF and BE, and the literal translation reads "Draw forward (extend) lines DA and DB until L and G." Here we see the sentence so similar to the one that pervades the 17th, 18th and 19th Century English textbooks that reads "produce lines DA and DB until L and G." Thus one possibility is that Adelard of Bath is responsible for introducing the error. However, it is known that in the 4th Century Theon of Alexandria's recension of the *Elements*, involved altering the language in some places and sometimes supplying alternative proofs. Furthermore, according to Busard [Bu83] all the manuscripts of the *Elements* known until the 19th Century were derived from Theon's text. Therefore it is possible that Theon is the culprit here. On the other hand Adelard of Bath translated his manuscript from the Al-Hajjaj manuscript in Arabic. Therefore one may wonder if Al-Hajjaj is to blame. However it is generally considered that the Arabic manuscripts are quite trustworthy and other Latin translations of Arabic manuscripts, such as that of Gerard of Cremona have a correct algorithm. Therefore the finger seems to point in the direction of Adelard of Bath.

7. The Problems of Language Translation

There are at least two ways in which a correct algorithm may, after some historical evolution, become an incorrect algorithm. A mathematician such as Theon of Alexandria, in writing a textbook may offer an alternate algorithm and if he does not understand the deep structure of the algorithm may substitute an incorrect one in its place. Another more likely event is that a translator (who may not even understand the shallow structure of the algorithm, or who may be totally dependent on the figure to make sense out of it) inadvertently gives an incorrect translated algorithm. It seems quite probable that a translator such as Adelard of Bath, looking at the figure, may have thought that a*ppending AE in a straight line with AD* is a rather clumsy way of stating that *AD should be extended*, and may have substituted the new phrasing without realizing that an ambiguity has been introduced. The reader may doubt that with such simple and elementary descriptions of algorithms as are found in Euclid's *Elements* a translator can become a traitor. An example will remove any doubt.

The most accurate and definitive Greek version of the *Elements*, the Heiberg edition, was translated in 1978 into French [Ka78]. The book [Ka78] contains, very conveniently, all the propositions in both Greek and French side by side. The introduction contains an interesting discussion on the problems of translation which we paraphrase in part.

> *It is no doubt easier to make a literal translation but such an attitude leads to serious inconveniences for understanding the text. The linguistic differences between Greek and French on one side and the evolution of the mathematical vocabulary on the other are liable to lead the reader into confusion. In the hope of presenting a directly accessible manuscript we have opted for a free translation remaining as true as possible to the text but attaching more importance to the spirit than the letter*

the following “trick” makes the essence “jump out of the page at you.” We fix the construction instead and for this fixed construction we “look” at all possible input configurations. The crucial part of Euclid’s construction (missing in Pedoe’s algorithm [Pe76] and missed by most of Euclid’s followers) is the *cone* determined by the rays DE and DF and making an angle of 60 degrees at D. This cone is *implicitly* constructed by the resulting concatenation of the equilateral triangle DAB and the extensions constructed in *Step 3* from A to F and B to E. This cone is as large as desired and always subtends 60 degrees. Therefore consider such a cone as fixed in space and refer to Fig. 6.1. Now point A must always lie on one ray DF. Also line segment BC must always have its endpoint B on the other ray DE. With the compass anchored on B Euclid’s construction first marks off a point G on BE such that BG equals BC. Then with the compass anchored on D it marks off a point L on AF such that DL equals DG. It is clear that for all possible configurations of points A and line segments BC the construction is valid. Variation in the distance between A and B does not change the essence of the proof. Furthermore, all possible relative positions of segment BC with respect to point A retain their property of cutting BE at G. It does not matter whether BC is greater than, less than or equal to AB. Neither does it matter if C lies on AB or DB or for that matter if it coincides with point A or D! Therefore the algorithm is well defined and executes in all possible cases. Since in all cases DB equals DA it follows that the algorithm yields the correct solution in all cases as well.

This then is the logic behind Euclid’s proof and, we might add that, Bertrand Russell [Ru51] and Dunham [Du90] not withstanding, it holds without the need of a figure. One wonders if Russell’s critique of Euclid is based on the ambiguous and/or incorrect algorithms written down by 19th century Oxford and Cambridge trained scholars such as [Sm1879], [HS1887] and [Ta1895] or either on the 12th century Latin manuscript of Gerard of Cremona or Peyrad’s pre-Theonian manuscript, where unambiguous and correct versions of Euclid’s second proposition appear that do not depend on a figure. We see at once Euclid’s brilliance in the extension of DA and DB in the directions of A and B to create the *cone* with apex at D rather than in the direction of D as done by Proclus for example. It is also easy to see with the aid of this *cone* that indeed there are no proper cases here at all. The cases fabricated and considered by Euclid’s commentators are artifacts of a lack of understanding of the underlying logic which, it is conjectured, Euclid had in mind when writing this construction and proof. In light of the culturally established belief held by so many that Euclid’s proofs only hold for certain cases together with the fact that almost all modern versions of the construction are either ambiguous or downright incorrect, it is easy to understand why Pedoe [Pe76] picks only one such case and claims to give Euclid’s original proof although it is missing the crucial construction of the *cone* mentioned above.

We close this section with a conjecture as to how it came about that so many of the 17th, 18th and 19th Century English textbooks contain an incorrect algorithm for Euclid’s second proposition. I believe the answer may lie in the famous Latin translation (of an Arabic manuscript by Al-Hajjaj) due to Adelard of Bath [Bu83].

Amongst the most well known medieval English translators of Euclid’s *Elements* was Adelard of Bath in the 12th Century. Actually Adelard of Bath’s name is associated with three distinct versions of the *Elements* and according to Busard [Bu83] it was version II “that became the most popular of the various translations of the *Elements* produced in the 12th Century and apparently the one most commonly studied in the schools.” Furthermore, this version is apparently the least authentic. In the words of Busard [Bu83] “not only are the enunciations differently expressed *structure* behind Euclid's proof.

First we should remember that when cases did in fact exist Euclid used figures to *illustrate* a construction and proof rather than make a *case* statement. In the words of Heath [He28]:

> "To distinguish a number of cases in this way was foreign to the really classical manner. Thus, as we shall see, Euclid's method is to give one case only, for choice the most difficult, leaving the reader to supply the rest for himself. Where there was a real distinction between cases, sufficient to necessitate a substantial difference in the proof, the practice was to give separate *enunciations* and proofs altogether."

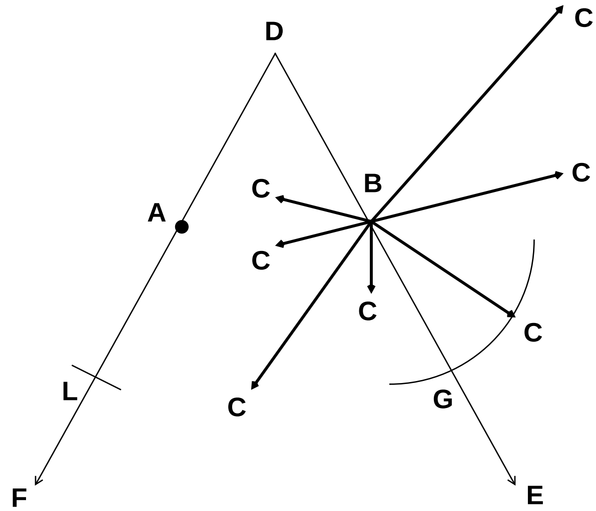

Fig. 6.1 Illustrating the proof of Euclid's *Proposition 2* for all cases.

This is indeed the social convention followed even today in computational geometry where the phrase "the remaining cases can be proved in a similar way" is seen in almost every published paper in the most scholarly of journals.

It is conjectured though that Euclid saw in *Proposition 2* no cases because fundamentally there aren't any. Furthermore, if the reader will follow through Euclid's original algorithm in all the possible "fabricated" cases enumerated in the previous section he or she will find that the algorithm is well defined in the modern sense and will execute correctly and terminate with the correct solution. Furthermore, the proof of correctness also follows through. This cannot be said of any of the subsequent algorithms and proofs offered by Heron, Proclus and the other Greek commentators of Euclid nor the 19th century English scholars. It should be mentioned here that one logical (out-of-context) situation consists of *Case 1.1.1* in which the point A lies at one endpoint of segment BC. Clearly in this pathological situation an equilateral triangle cannot be constructed on AB and the algorithm would be undefined and fail to execute. However, the context of the situation, i.e., the purpose of the problem is *to transfer a distance*. If A coincides with either B or C then there is no *transfer* of distance, the problem does not exist, or if you like, the answer, namely segment BC, is already known at the start. Therefore, the algorithm is clearly intended to work for all points A on the plane except B and C.

The reader may experience an interesting effect upon actually carrying out Euclid's construction and proof for all the cases enumerated above, and that is the *Eureka* experience in which the *essence*, *semantics*, or *deep structure* behind Euclid's construction is made manifest. Once this happens it is transparently clear that Euclid's algorithm and proof of correctness are valid for all cases one could possibly imagine. In fact in the view expressed here it becomes clear that fundamentally there are indeed no cases.

It is difficult to grasp the essence of the algorithm-proof by fixing an input configuration and then analyzing variations in constructions as in the work of the Greek commentators. However between B and C.

Some of the above cases (but certainly not all!) were discussed by the Greek commentators and are included in the work of Proclus [Mo70]. Usually a proof that Euclid's algorithm worked correctly was then provided for the particular case at hand. Sometimes the actual *algorithm* given by Euclid was changed to handle the special case. For example, for a particular input configuration in *Case 2.1* with the distance between A and B less than the distance between B and C, Proclus objects to Euclid's algorithm because line segment BC "gets in the way" of the construction of triangle ABD above segment AB (see Fig. 5.1). In the words of Proclus, "for there is not room." Heath [He28] notes that Heron of Alexandria circa 100 A.D. in his commentary on the *Elements* also sometimes used constructions different from Euclid's to circumvent objections of this type. The algorithm of Proclus [Mo70] for this particular case follows (see Fig. 5.1).

Begin

Step 1: Let a circle be drawn with center at B and distance BC.

Step 2: Let the lines AD and BD be produced to F and G.

Step 3: With centre at D and distance DG let the circle GE be described.

[Exit with AE as the solution.]

End

Note how Proclus has changed the clear line-extension statements of Euclid's algorithm to the ambiguous statements (Let the lines AD and BD be produced to F and G.) found in the 19th century accounts and that the correctness of the construction is made to depend on the figure!

Another fascinating manuscript is an Arabic book titled *On the Resolution of Doubts in Euclid's Elements and Interpretation of Its Special Meanings* written in 1041 A.D. by Ibn al-Haytham. A copy of this book made in 1084 A.D. was found in the University of Istanbul Library [Ha1041]. As the title suggests this is not a translation of the *Elements* but a discussion about well known criticisms of Euclid's work. In discussing Euclid's second proposition al-Haytham discusses four basic cases in terms of the type of input: (1) point A is either B or C, (2) A lies on the line segment BC, (3) A lies on the line passing through BC, and (4) A lies outside the line passing through BC. In addition to these he has a very strange case that does not appear to have been mentioned anywhere else and this is the case when the line segment BC and the point A are separated by a valley or a river so that the line joining the points A and B cannot be drawn! His solution to this last case is most puzzling, for he writes that the way to handle this case is to measure the line segment and redraw it in the neighborhood of the point, after which Euclid's procedure is then applied! It would appear that Ibn al-Haytham was not lacking a sense of humor in his mathematical writings.

6. Euclid's Algorithm Reconsidered

It is clear from the above discussion that Euclid's followers were concerned that perhaps Euclid's algorithm and proof of correctness did not hold for all possible configurations of the input to the problem. I will argue that the commentators themselves succumbed to the fallacy of "going by the figure" even more than Euclid himself and that they missed the *essence*, *semantics* or *deep* algorithm is designed to work only for inputs in *general position*, it should also be able to handle singularities such as when point A lies on the segment BC or A is equidistant from B and C. Similarly, a proof of correctness must establish that in all situations the algorithm will yield the correct solution. Euclid had the habit, as is well illustrated by Fig. 4.1, of including only one figure to illustrate the construction and proof. It is only natural that a reader may thus wonder on stepping through the algorithm on the given figure whether the same steps would work on a completely different figure. The same reader may even be skeptical as to whether the arguments in the proof of correctness would carry over with the same letters used as labels of crucial points derived during the construction. This in fact appears to have been the reaction of early Greek commentators of the *Elements* who criticized Euclid for leaving out cases that they discovered missing and then supplied accompanying proofs of their own. An in depth commentary of Euclid's elements and subsequent criticisms made against it was written down in the 5th century by Proclus [Mo70]. Proclus himself does not usually criticize Euclid and on several occasions actually comes to his defense. In the words of Glenn Morrow [Mo70]:

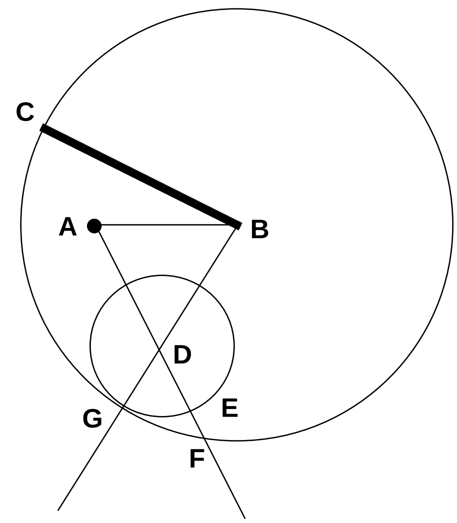

Fig. 5.1 Proclus's figure for the proof of a subcase of *Case 2.1 of Proposition 2*.

> "When in the proof of a theorem Euclid uses only one of two or more possible cases, as is his custom, Proclus will often prove one or more of the omitted cases; sometimes he simply calls attention to them and recommends that his readers, "for the sake of practice," prove them for themselves. Sometimes he gives an alternative proof of a theorem devised by one of his predecessors for the obvious purposes of showing the superior elegance or appropriateness of Euclid's demonstration."

It is instructive to illustrate some of the objections that early Greek commentators had and how they resolved them. First we note that indeed one can conjure up many special cases of an initial configuration of point A and line segment BC. For example: *Case 1*: A may lie on the line collinear with BC or *Case 2*: A may lie on one side of the line collinear with BC. In *Case 1* A may lie on the line segment BC (*Case 1.1*) or off the line segment BC (*Case 1*.2). If A lies on BC then in *Case 1.1.1* it may lie on an endpoint of BC or in *Case 1.1.2* on the interior of BC, and in the latter case we have two more cases depending on whether A is closer to B or closer to C. In *Case 1.2* where A lies off segment BC, A could be closer to B (*Case 1.2.1*) or to C (*Case 1.2.2*). Furthermore, *Case 1.2.1* divides into two more cases depending on whether the distance between A and B is greater than or less than the distance between B and C. *Case 2* in which A lies off the line collinear with BC can also be divided into cases using a variety of criteria. For example we might consider two cases depending on whether the line segment BC lies in the interior (*Case 2.1*) or the exterior (*Case 2*.2) angle that triangle ABD makes at D. Finally each of these two cases determine two more cases depending on whether the distance between A and B is greater than or less than the distance ing Peyrard enough time to finish his translation [Pe1819]. In Peyrard's manuscript which he emphasizes is a literal translation, the crucial *Step 3* is written as "Menons *les droites AE, BZ dans la direction de DA, DB,"* and is thus in agreement with **Algorithm Euclid** described above.

5. Cases in Constructions and Proofs

The above discussion brings up naturally the general question of the analysis of *cases* in Euclid's constructions, modern computational geometry, and geometric proofs in general. We should note here that when we talk about *cases* today we generally mean equivalence classes of input configurations rather than instances of the construction sequence resulting from a set of choices made as a result of the ambiguities of the algorithm's description, as are the case classifications in Taylor [Ta1895] and Lardner [La1861]. An algorithm must be specified unambiguously and should execute correctly for all inputs it was designed to handle. Much criticism has been heaped on Euclid over the past two thousand years for his alleged sloppiness in his constructions and proofs concerning the question of cases. For one thing he has been blamed of giving proofs of correctness that depend severely on the figure accompanying the proof. For example according to Bertrand Russell [Ru51]:

> "A valid proof retains its demonstrative force when no figure is drawn, but very many of Euclid's earlier proofs fail before this test... The value of his work as a masterpiece of logic has been very grossly exaggerated."

Again, in the words of William Dunham [Du90]:

> "Admittedly, when he allowed himself to be led by the diagram and not the logic behind it, Euclid committed what we might call a sin of omission. Yet nowhere in all 465 propositions did he fall into a sin of commission. None of his 465 theorems is false."

Finally, in the words of Felix Klein [Kl39]:

> "Euclid... always had to consider different cases with the aid of figures. Since he placed so little importance upon correct geometric drawing, there is real danger that a pupil of Euclid may, because of a falsely drawn figure, come to a *false* conclusion."

A proposition that has a plethora of cases and that has been the subject of much criticism of Euclid is in fact *Proposition 2*, the topic of this paper. It will be argued here using this proposition as a "case" study that much of the criticism of Euclid regarding case analysis stems from a lack of deep understanding of his original work due in part to the writings of the early Greek commentators of the *Elements* such as Heron and Theon of Alexandria and others reviewed by Proclus [Mo70] in the 5th century and exacerbated by a 12th century Latin translation of an Arabic manuscript by Adelard of Bath [Bu83] and many English scholars of the 19th century. Furthermore, if we judge the original algorithm and proof of correctness of Euclid's *Proposition 2* using today's highest standards in the field of computational geometry Euclid deserves praise for his brilliance.

Consider then Euclid's second proposition: *To place at a given point* (*as an extremity*) *a straight line equal to a given straight line.* Clearly an algorithm for carrying out this task has to execute, i.e., be well defined for all inputs, i.e., for all possible line segments BC and all points A no matter how they are positioned with respect to each other in the plane. Furthermore, unless the texts where it is asked to extend DA and DB the phrase “in *a straight line with* DA, DB” is absent because the direction is obviously implied by the extension of the sides of the equilateral triangle. In **Algorithm Euclid** on the other hand extensions emanating from A and B may do so in any direction and thus the phrase “in *a straight line with* DA, DB” is provided for precision. The skeptical reader may nevertheless at first glance insist that the straight line AE may be produced in a straight line with DA in such a way that D lies in AE (in other words, AE emanating from A in the opposite direction as that shown in Fig. 4.1). However, it is obvious that if this were the case intended, Euclid would have used the phrase “Let *the straight line DE be produced in a straight line with AD.” N*o room is left here for choosing the direction of the extensions of DA and DB as is clearly the case in **Algorithm 19C**.

At this point one may wonder about the authenticity and correctness of the accounts of Heiberg [He1883], Heath [He28], and Dijksterhuis [Di55]. Here we should point out that the Greek text by Heiberg is considered to be the *definitive* edition. It consists of portions taken from different Greek manuscripts spanning the 9th to 12th centuries and considered by philologists to be the most authentic. There also exist several interesting Latin manuscripts which are translations of Arabic manuscripts. Perusal of the first printed edition of the Latin translation of the Arabic (Ishaq-Thabit) version of Euclid’s *Elements* believed to be made by the monk Gerard of Cremona in Toledo during the 12th century [Bu84] following its discovery in Bagdad, would also appear to be more convincing than examining English texts of the 17th, 18th and 19th centuries. Indeed, apart from the fact that the letters E and F in Heath [He28] are absent in [Bu84] and their role subsumed by L and G, respectively, the algorithms and proofs of correctness found in [He1883], [He28], and [Di55] are identical to that found in the 12th century Latin manuscript. This 12th century algorithm is a Latin translation of an Arabic translation of a Theonian Greek manuscript. In fact all Arabic translations are believed to descend from the 4th century recension by Theon of Alexandria. Anyone who has played the translation game may wonder how this version compares with early Greek manuscripts with respect to the crucial *Step 3* which states (referring to Fig. 4.1) “Let *the straight lines AE, BF be produced in a straight line with DA, DB.” I*n another 12th century Sicilian Latin translation (of unknown authorship) of Euclid’s *Elements* made *directly* from the Greek [Bu87] *Step 3* is stated as follows:

“Educantur *in directo rectis DA et DB recte AE et BF.”*

This translates to “Lead *forth the straight lines AE and BF in a straight line with (in the direction of) the straight lines DA and DB”* and is thus in agreement with the Gerard of Cremona version and Heiberg’s definitive edition.

As a final piece of evidence that **Algorithm Euclid** described above is indeed Euclid’s original algorithm we consider the so-called manuscript P, the Vatican manuscript No. 190. Until 1804 all manuscripts of Euclid’s *Elements* were believed to be descended from Theon’s 4th century recension. When Napoleon conquered Italy he stole from the Vatican a Greek manuscript (No. 190) of Euclid’s *Elements* which he kept in the King’s Library in Paris. F. Peyrard, a professor at the Lycee Bonaparte, wanted to write a definitive French version of the *Elements* using the best Greek manuscripts at his disposal and towards that end obtained access to the King’s Library. There he found manuscript No. 190 and to his astonishment discovered he had in his hands a pre-Theonian 10th century manuscript. In the mean time the Allied Forces defeated Napoleon and forced France to return all stolen works of art. On the request of the French government the Pope made Peyrard a happy man by granting an extension of the return date of the manuscript thus givrithm went up in smoke.

In spite of the criticism often directed at Euclid, one may find it difficult to believe that he could have been guilty of an oversight such as that suggested by Pedoe's version of his algorithm. On the other hand the consensus of descriptions exemplified by **Algorithm 19C** given by English scholars such as Lardner [La1861], Todhunter [To1876], Smith [Sm1879], and Hall and Stevens [HS1887] as well as **Algorithm T** of Taylor [Ta1895] and those of the 18th and 17th centuries may make the reader wonder whether Euclid's original algorithm did suffer from similar defects. However, established authorities on Euclid such as Heiberg [He1883], Heath [He28] and Dijksterhuis [Di55], have an algorithm significantly different from the ones thus far described. The figure in these three works is given in Fig. 4.1. and the algorithm is given below. We omit the proof of correctness as it is exactly the same as that given by Pedoe.

Proposition 2: *To place at a given point* (*as an extremity*) *a straight line equal to a given straight line.*

Algorithm Euclid: [Heath's version as well as the original version according the 12th century manuscript of Gerard of Cremona]

Input: Let A be the given point, and BC the given straight line. {Thus it is required to place at the point A (as an extremity) a straight line equal to the given straight line BC.} Fig. 4.1.

Begin

Step 1: From the point A to the point B let the straight line AB be joined.

Step 2: On AB [using **Algorithm 1**] let the equilateral triangle DAB be constructed.

Step 3: Let the straight lines AE, BF be produced in a straight line with DA, DB.

Step 4: With centre B and distance BC let the circle CGH be described.

Step 5: With centre D and distance DG let the circle GKL be described.

Exit with AL as the solution.

End

Note that in Fig. 4.1 the length of BC is indeed larger than the distance between A and B and Pedoe's version of Euclid's algorithm would not work in this case. However, for a reason mysterious (I will offer a conjecture later) Pedoe leaves out *Step 3* in the above version of Euclid's algorithm. This crucial step in Euclid's algorithm constructs the extensions of DA and DB in directions E and F, respectively, thus ensuring that whether or not the length of BC is larger than the distance between A and B, the algorithm continues to "execute" and the figure remains the same in the sense that point G exists and lies on BF. Note the significant difference between the manner in which DA and DB are to be produced in **Algorithm Euclid** as compared to **Algorithm 19C** and **Algorithm T**. In the latter two algorithms the ambiguous instructions state that the *sides of the equilateral triangle* DA *and* DB *are to be produced.* In **Algorithm Euclid** on the other hand the statement in Step 3 concerning the extension of DA and DB is unambiguous. It states: "*Let the straight lines* AE, BF *be produced in a straight line with* DA, DB." In other words it specifies that (1) the extensions are to emanate from A and B (the endpoints of the base of triangle DAB) and (2) they should be collinear with (in a straight line with or in the direction of) DA and DB. In all other tinction (according to the claim on its front page) of containing the first English translation from Latin. This appears to contradict the belief that the first English translation of Euclid's *Elements* to be printed was translated by H. Billingsley, printed by the famous English printer John Day and issued in 1570 [Ar50], [Sh28]. What is worth noting about the algorithms in all three of these texts, however, is that 1) they are all identical to each other, 2) *like* **Algorithm 19C**, they are ambiguous but, 3) *unlike* all other algorithms I have encountered, they begin not by connecting point A to one of the endpoints of segment BC but by constructing a circle of radius BC centered at one of the endpoints of BC. Then in the second step point A is joined to the endpoint selected as the center in the previous step. Note that this ordering circumvents the problem that **Algorithm T** has with *Steps 1* and *2* and furthermore allows us to ignore Lardner's caveat intended to resolve it.

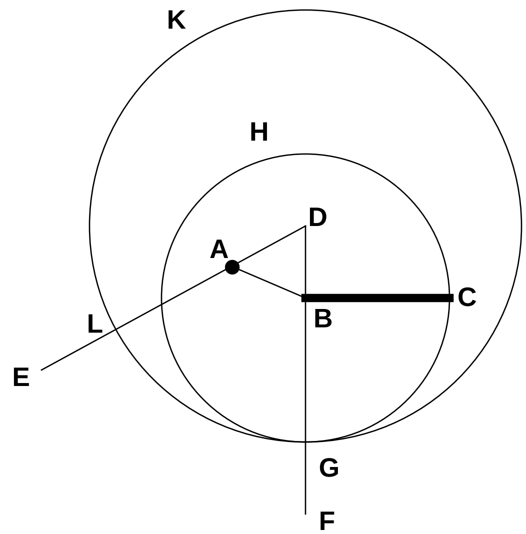

Fig. 4.1 Euclid's figure for the proof of *Proposition 2* according to Heath.

4. Euclid's Construction According to Gerard of Cremona and Peyrard

One is naturally led to the question: which of all these algorithms is the one Euclid originally proposed? It would be easy to answer this question by looking up Euclid's original manuscript. Unfortunately history has made this impossible. In the year 332 B.C. Alexander the Great, at the age of 24, conquered Egypt and founded the city of Alexandria. When, after conquering the rest of the world, he died at the age of 33 in Babylon (just south of present day Bagdad) his generals divided up the world into pieces amongst each other. In this way Egypt fell into the hands of Ptolemy I in 306 B.C. Ptolemy II created the University of Alexandria which became by virtue of its excellent scholars (including Euclid) and its impressive library (three quarters of a million books including Euclid's original version of *The Elements*) the intellectual and scientific center of the world. In 48 B.C. Julius Caesar occupied Alexandria and intended to carry a large portion of the library with him back to Rome. The academic community held a demostration which was quickly quelled by Caesar's army. In the ensuing fighting Caesar set fire to the Egyptian fleet in the Great Harbour. The fire spread to wharehouses on the docks and from there to the library at which time many of the books were burned. More books were burned during later Egyptian revolts, one in 272 A.D. quelled by the Emperor Aurelianus and another in 295 A.D. quelled by the Emperor Diocletian. In the 4th and 5th centuries the *polytically-correct-thinking* movement, with Christianity as the governmental dogma, became paramount in Alexandria and zealous Christian bishops began to persecute the pagan writers (mathematicians) and their books. Bishop Theophilus in 391 A.D. lead a Christian mob and destroyed the Temple of Serapis which housed many of the remaining books. The last mathematician alive in Alexandria, a woman by the name of Hypatia and daughter of Theon was torn limb from limb in the streets of Alexandria by an enraged mob led by Bishop Cyril [La41]. Finally, the Arabs invaded Egypt in 646 A.D. and General Amr ibn-al-As burned the remaining books allegedly because [Be71] "if the books agreed with the Koran they were superfluous; if they disagreed they were pernicious." In short, in all likelihood Euclid's original algoneed not be produced to G. According to the algorithm therefore the solution is given by AG' which is clearly incorrect since AG' is smaller than AB whereas BC is greater than AB, by assumption. Therefore although the ambiguities of **Algorithm 19C** have been removed by Taylor, **Algorithm T** does not always yield the correct solution on a given *line-point* configuration depending on which construction strategy is applied. Furthermore, **Algorithm T** suffers from an additional minor bug not even present in **Algorithm 19C**. Notice that *Step 1* in **Algorithm 19C** does not offer choice. However **Algorithm T** asks that A be connected to one of the extremities of BC, one that we are free to choose. However, if we choose to connect A to C (rather than B as in Taylor's figure) then it is impossible to execute *Step 2* and the algorithm crashes.

Another author, Lardner [La1861], also follows his presentation of an ambiguous algorithm identical to **Algorithm T** with a discussion of how the student should be careful about different cases arising from the varieties of different input configurations. In his own words:

> "The different positions which the given right line and the given point may have with respect to each other, are apt to occasion such changes in the diagram as to lead the student into error in the execution of the construction for the solution of this problem.
>
> Hence it is necessary that in solving this problem the student should be guided by certain *general* directions, which are independent of any particular arrangement which the several lines concerned in the solution may assume. If the student is governed by the following general directions, no change which the diagram can undergo will mislead him."

Lardner then proceeds to present six general rules concerning what can and cannot be done in order to ensure that **Algorithm T** works correctly on all inputs. This discussion includes a case analysis of construction strategies and, unlike Taylor [Ta1895], does not allow DA and DB to be extended in either direction but insists that they be extended through the base of the constructed triangle thus concluding that the solution to Euclid's second proposition has *four* cases rather than Taylor's *eight*. Another general rule, that Lardner insists should be followed, is that the center of the circle constructed in *Step 3* should lie at the extremity of BC connected to A in *Step 1*, thus avoiding one of Taylor's problems.

Another variation occurs in a much earlier Scottish book on Euclidean geometry published in 1831 by John Playfair [Pl1831] which has a variation of **Algorithm 19C**. In this book we are asked to extend DA and DB to E and F respectively and thus the ambiguity of **Algorithm 19C** is also present here. However, unlike **Algorithm 19C** or **Algorithm T** the algorithm in [Pl1831] first performs the extensions and subsequently constructs the circles.

We close this section with a note on text books of the 18th and 17th centuries. In these two centuries combined the number of editions of Euclid's *Elements* published was less than half of the number for the 19th century, about 325 and 280 in the 18th and 17th centuries, respectively. It is also much more difficult to find copies of these earlier editions. I have held in my hands only two editions from the 18th century [Wi1703], [Ba1705] and one from the 17th century [Cl1654], having found all three in the special collection of the library at Queens University in Kingston, Ontario. The 1705 manuscript by Issac Barrow (from Trinity College, Cambridge) has the additional disthe point A a straight line equal BC.} Refer to Fig. 3.2.

Begin

Step 1: Draw AB, the straight line from A to one of the extremities of BC.

Step 2: On it construct an equilateral triangle DAB.

Step 3: With B as centre and BC as radius, describe the circle CEF, meeting DB (produced **if** necessary) at E.

Step 4: With D as centre and DE as radius, describe the circle EGH, meeting DA (produced **if** necessary) at G.

Then AG is the straight line as required.

End

Note that Taylor is careful to add in *Steps 3* and *4* the explicit ***if** statements* that DB and DA are to be produced if necessary. Therefore we presume that if the construction circle CEF intersects the sides of equilateral triangle ABD then the extension of DA need not be carried out. Unlike the previous 19th century geometry books Taylor follows the proof of *Proposition 2* with the following interesting discussion.

> "It is assumed in this proposition that the straight line DB intersects the circle CEF. It is easily seen that it must intersect in two places.
>
> It will be noticed that in the construction of this proposition there are several steps at which a choice of two alternatives is afforded: (1) we can draw either AB or AC as the straight line on which to construct an equilateral triangle: (2) we can construct an equilateral triangle on either side of AB: (3) if DB cut the circle in E and I, we can choose either DE or DI as the radius of the circle which we describe with D as centre.
>
> There are therefore three steps in the construction, at each of which there is a choice of two alternatives: the total number of solutions of the problem is therefore 2x2x2 or eight."

We see that Taylor's way of dealing with the ambiguities discussed above is to explicitly acknowledge that there are eight different cases to Euclid's proposition that depend on how the construction is carried out, that we are free to choose any one of these eight paths through the implied decision-tree, and that the sides DB and DA need not be produced if not necessary. In light of this classification let us follow down one path of these choices on the input configuration illustrated in Fig. 3.2 where it assumed that the length of CB is greater than the length of CA. In our first choice we therefore select AB as the segment on which to construct our equilateral triangle. Our second decision will be to construct the triangle on the side shown in Fig. 3.2. Now since the circle CEF does not intersect the triangle we extend DB which cuts the circle at the two points E and I. According to Taylor we may now choose either DE or DI as the radius of the circle which we describe with D as centre. Accordingly let us choose DI. Now, this circle with D as centre intersects DA at G' playing the role of G in his algorithm, and therefore, according to *Step 4*, DA the point A a straight line equal BC.} See Fig. 3.1.

Begin

Step 1: Join AB.

Step 2: On AB describe an equilateral triangle DAB.

Step 3: From centre B, with radius BC, describe the circle CGH.

Step 4: Produce DB to meet the circle CGH at G.

Step 5: From centre D, with radius DG, describe the circle GKF.

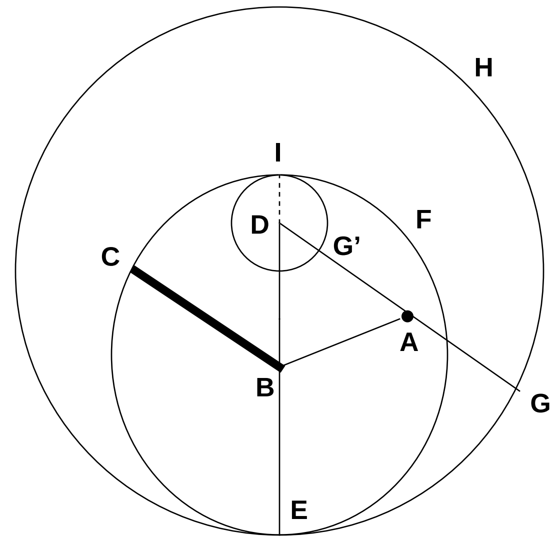

Fig. 3.2 Illustrating Taylor’s version of Euclid’s *proposition 2.*

Step 6: Produce DA to meet the circle GKF at F.

Then AF shall be equal to BC.

End

This algorithm is certainly an improvement over Pedoe’s algorithm as it appears to work correctly for some input configurations whether BC is greater than or less than BA. Nevertheless the algorithm suffers from ambiguous statements. *Step 4* asks us to produce (extend in length) DB to meet the circle CGH at G but it does not tell us in which direction (emerging from D or from B) to produce DB and certainly in either direction we are bound to meet the corresponding circle constructed in *Step 3*. Fig. 3.1 shows only one possible case but had we produced DB in the direction from B to D instead of the direction shown we would have obtained a completely different intersection point G. A similar problem exists with *Step 6*.

The ambiguities observed in the algorithms described in [Sm1879], and [HS1887] which are exemplified here as **Algorithm 19C** are absent in the exposition by Taylor [Ta1895], if not in the body of the algorithm at least in the subsequent discussion where it is indicated that we are free to choose one or the other alternative as is *Step 1.* It is therefore instructive to examine his algorithm and accompanying discussion in more detail.

Proposition 2: *From a given point to draw a straight line equal to a given straight line.*

Algorithm T: [Taylor’s version]

Input: Let A be the given point, and BC the given straight line. {It is required to draw from

BC. Being what it was required to do.

End of proof.

We remark here that Pedoe's figure, shown in Fig. 2.2, is considerably different from those in other sources on Euclid such as Heiberg [He1883], Heath [He28] and Dijksterhuis [Di55] for example. Much more serious, however, is the fact that **Algorithm P** given by Pedoe is incorrect! It is clear that for a solution to be obtained by **Algorithm P** it is crucial that the circle centered at B with radius BC intersect DB at G. Otherwise G is undefined and the rest of the algorithm makes no sense. Now consider what happens when the length of BC is greater than the distance from A to B. Clearly the circle centered at B with radius BC will completely enclose triangle ABD in its interior and the construction fails! In modern parlance the algorithm is not well defined for such an input and the algorithm crashes.

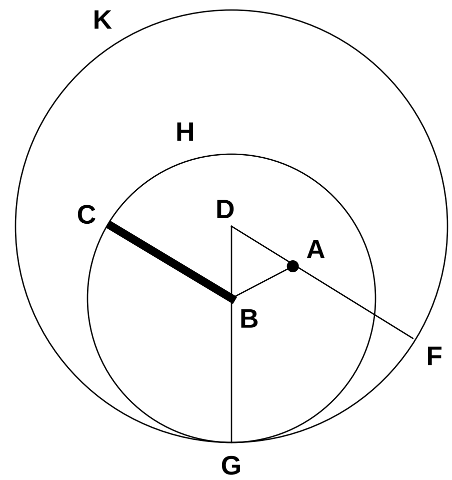

Fig. 3.1 Popular 19th century figure for the proof of Euclid's *Proposition 2.*

3. Euclid's Construction According to 19th, 18th and 17th Century Scholars

During the 19th century (which witnessed a total of more than 700 editions of *The Elements* published) there existed a flurry of activity in England with regards to the writing of text-books on the topic of Euclid's *Elements* for use in the schools and colleges. A sample of several of these books [La1861], [To1876], [Sm1879], and [HS1887], yields a common (apart from some trivial notational differences) algorithm and illustrative figure for Euclid's second proposition. However, both the algorithm and figure are quite different from Pedoe's [Pe76]. Consider then the algorithm according to one of these sources [HS1887].

Proposition 2: *From a given point to draw a straight line equal to a given straight line.*

Algorithm 19C: [Popular 19th Century version]

Input: Let A be the given point, and BC the given straight line. {It is required to draw from science: *subroutines*. In the algorithm of his second proposition described next he uses **Algorithm 1**. Below we give Pedoe's description of Euclid's construction.

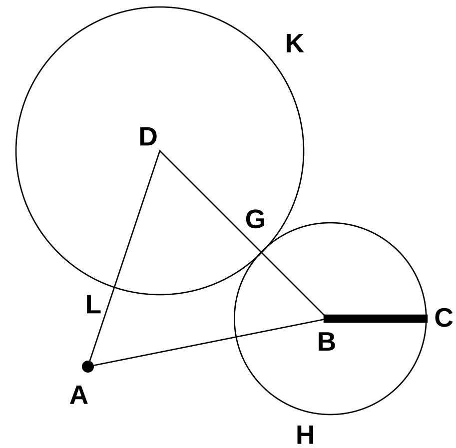

Fig. 2.2 Pedoe's figure for proving Euclid's *Proposition 2* .

Proposition 2: *To place at a given point* (*as an extremity*) *a straight line equal to a given straight line*.

Algorithm P: [Pedoe's version]

Input: Let A be the given point, and BC the given straight line. {Thus it is required to place at the point A (as an extremity) a straight line equal to the given straight line BC.} Fig. 2.2.

Begin

Step 1: From the point A to the point B let the straight line AB be joined.

Step 2: On AB [using **Algorithm 1**] let the equilateral triangle DAB be constructed.

Step 3: With centre B and distance BC let the circle CGH be described.

Step 4: With centre D and distance DG let the circle GKL be described.

Exit with AL as the solution.

End

Proof of correctness: Then, since the point B is the centre of the circle CGH, BC is equal to BG.

Again, since the point D is the centre of the circle GKL, DL is equal to DG, and in these DA is equal to DB.

Therefore the remainder AL is equal to the remainder BG.

But BC was also proved equal to BG. Therefore each of the straight lines AL, BC is equal to BG.

And things which are equal to the same thing are also equal to one another.

Therefore AL is also equal to BC.

Therefore at the given point A the straight line AL is placed equal to the given straight line eral triangle on the straight line AB.} See Fig. 2.1.

Begin

Step 1: With centre A and distance AB let the circle BCD be described.

Step 2: With centre B and distance BA let the circle ACE be described.

Step 3: From the point C, in which the circles cut one another, to the points A, B let the straight lines CA, CB be joined.

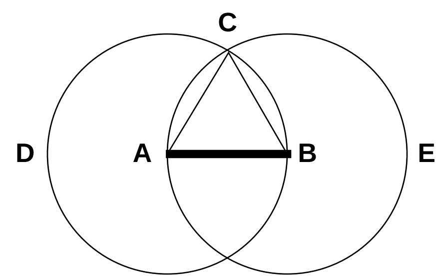

Fig. 2.1 Euclid’s figure for the proof of *Proposition 1*.

End

Proof of Correctness:

Now, since the point A is the centre of the circle CDB, AC is equal to AB.

Again, since the point B is the centre of the circle CAE, BC is equal to BA.

But CA was also proved equal to AB; therefore each of the straight lines CA, CB is equal to AB.

And things which are equal to the same thing are also equal to one another; therefore CA is also equal to CB.

Therefore the three straight lines CA, AB, BC are equal to one another.

Therefore the triangle ABC is equilateral; and it has been constructed on the given finite straight line AB. Being what it was required to do.

End of Proof

Of course neither Euclid nor Pedoe use the words *algorithm*, *input*, *begin* and *end. N*either do they use the phrases *proof of correctness* nor *end of proof*, nor do they label separate chunks of the algorithm with the word *Step* such-and-such. However early Latin manuscipts do preface the construction by the words *exempli causa* and the proof by *probatio eius*. We include these well known terms found in modern computer science for clarity of layout and to delineate that these divisions did appear in essence in at least the earliest Arab and Latin translations of Euclid’s *Elements*. The important thing is that Euclid always gave the algorithm first and the arguments to substantiate the correctness of the algorithm immediately afterwards. Even today too many writers still publish geometric algorithms without including a proof of correctness in spite of the many geometric algorithms that have been found to be incorrect [To84]. These authors could certainly take a lesson here from Euclid. Sometimes, as we shall see below, the algorithms in the *Elements* include unnecessary steps for obtaining the solution but these steps have the benefit of simplifying the ensuing proof of correctness. Euclid also made use of another common practice in modern computer space to some other location to draw a circle with the chosen radius. This operation cannot be done with a collapsing compass. The collapsing compass is, like the other machines, an *idealized* machine which allows the compass to be opened to a chosen radius and a circle drawn, but no distance can be *transferred*. It is as if when the compass is lifted off the work-space it collapses and thus erases any trace of the previous aperture made. Of course more complicated machines can be obtained by combining sets of simple machines. For example in Euclid's *Elements* he uses the *straight edge* and *collapsing compass* (the combination of machines 1 and 3) as his model of computation. Attempts have also been made to specify the primitive operations allowed with each type of machine [Le02] and to design constructions that require fewer operations than did Euclid's original constructions. Another active area of research has been to analyze and compare different machine models in terms of their computational power [Ho70], [CR81], [Av87], [Av90]. For example, in 1672 Georg Mohr [Mo1672] and in 1797 the Italian geometer Lorenzo Mascheroni [Ma1797] independently proved that any construction that can be carried out with a straight edge and a compass can be carried out with a compass alone and Jacob Steiner proved in 1833 that the straight edge is equivalent in power to the compass if the former is afforded the use of the compass once [SA48]. To remind the reader that the *straight edge* and *compass* are not yet obsolete computers we should point out that the Mohr-Mascheroni result was strengthened as recently as in 1987 by Arnon Avron [Av87] at the University of Tel Aviv.

The earliest proof of the equivalence of models of computation is due to Euclid in his second proposition of Book I of the *Elements* in which he establishes that the *collapsing compass* is equivalent in power to the *compass*. Therefore in the theory of equivalence of the power of models of computation, Euclid's second proposition enjoys a singular place. However, like much of Euclid's work and particularly his constructions involving many cases, his second proposition has received a great deal of criticism over the centuries. In this paper it is argued that it is Euclid's commentators and translators that are at fault and that Euclid's original algorithm and proof are beyond reproach. Since this proposition uses *Proposition 1* to obtain a solution we begin by outlining the latter.

2. Euclid's First Two Propositions According to Pedoe

Pedoe [Pe76] contains a lively discussion of Euclid's elements of geometry applied to painting, sculpture and architecture throughout recent history and to illustrate Euclid's method he presents the first two propositions of Book 1 of his *Elements*. Earlier in the book he actually has a completely different algorithm and proof of *Proposition 2* to which we shall return at the end of this paper. However, at this later point in the book he states that "it *is of interest to read how it appears in Euclid.*" Subsequently the following algorithms and proofs of correctness are presented.

Proposition 1: *On a given finite straight line to construct an equilateral triangle.*

Algorithm 1:

Input: Let AB be the given finite straight line. {Thus it is required to construct an equilat-

A NEW LOOK AT EUCLID'S SECOND PROPOSITION

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Abstract

There has been considerable interest during the past 2300 years in comparing different models of geometric computation in terms of their computing power. One of the most well known results is Mohr's proof in 1672 that all constructions that can be executed with *straight-edge* and *compass* can be carried out with *compass* alone. The earliest such proof of the equivalence of models of computation is due to Euclid in his second proposition of Book I of the *Elements* in which he establishes that the *collapsing compass* is equivalent in power to the *modern compass*. Therefore in the theory of equivalence of models of computation Euclid's second proposition enjoys a singular place. However, like much of Euclid's work and particularly his constructions involving cases, his second proposition has received a great deal of criticism over the centuries. Here it is argued that it is Euclid's early Greek commentators and more recent expositors and translators that are at fault and that Euclid's original algorithm, according to Gerard of Cremona's Latin translation of a 12th century Arabic manuscript as well as Peyrard's French translation of a pre-Theonian 10th century Greek manuscript, is beyond reproach.

1. Introduction

In the modern comparative study of geometric algorithms it is imperative to first define the *models of computation*, i.e., the characteristics of the machine that will execute the algorithms [PS85]. A model of computation specifies the number of *processors* used, whether they are used *sequentially* or in *parallel*, the primitive operations allowed and the cost associated with each of these operations. For example, one of the preferred conceptually abstract models or *ideal* machines in which many geometric algorithms are compared is the *Real RAM* (Random Access Machine [AHU74]) in which each unit of storage space is capable of holding a real number of unlimited precision and in which the primitive operations that can be performed in one unit of time include the arithmetic operations consisting of addition, subtraction, multiplication and division, comparisons between real numbers, reading from and writing into a storage location as well as perhaps some more powerful operations such as computing *kth* roots, trigonometric functions and other analytic functions. More controversial assumptions sometimes include the *ceiling* and *floor* functions.

In classical constructive geometry mathematicians have also been concerned with defining the *models of computation*, i.e., the characteristics of the "machine" that will execute the algorithms. Typical machines that have been used in the past starting with Euclid include 1) the *straight edge*, 2) the *ruler*, 3) the *collapsing compass*, 4) the *compass*, 5) the *fixed-aperture compass*, 6) the compass with aperture *bounded from above*, and 7) the compass with aperture *bounded from below* just to name a few [Sm61], [Ho70], [CR81], [Ko86]. The *collapsing compass* deserves elaboration here. With the regular compass one can open it, lock it at a chosen aperture and lift it off the work-