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$[p_i, p_j]$ lies in P. It is an open question whether this decomposition can be computed in $o(n^2)$ time and neither is a super-linear lower bound known for this problem. However, under a slightly different visibility constraint Su and Chang [SC91a] are able to obtain an O(n log n) time algorithm for computing the RNG of a set of line segments. Clearly a simple polygon is a special case of a set of line segments and hence under their visibility constraint the RND of a simple polygon can be computed in O(n log n) time. Pankaj Agarwal has shown that their methods will also yield an O(n log n) algorithm for the RND as define by ElGindy and Toussaint[ET88].

1.2.2 Special classes of polygons

The fastest known algorithm [ET88] for computing the RND of a simple polygon is $O(n^2)$. On the other hand, for *convex* polygons the RND can be computed in O(n) time [Su83], and so can the Delaunay triangulation [AGSS]. However, it is shown in [ART87] that O(n log n) is a lower bound for computing the Delaunay triangulation on the vertices of a *star-shaped* or *monotone* polygon. It is unknown whether any other proximity graphs can be computed in linear time for the case of convex polygons. Furthermore, for most proximity graphs it is unknown whether they can be computed in $o(n^2)$ time for special classes of simple polygons such as *star-shaped*, *monotone* or *unimodal* polygons. For *unimodal* polygons the RNG and MST can be computed in O(n) time [Ol89]. It is unknown whether the Delaunay triangulation on the vertices of a *unimodal* polygon can be computed in linear time.

2. Recognizing Proximity Graphs

One area as yet almost totally unexplored concerns the question of the recognition of proximity graphs. The only known result concerns Delaunay triangulations. Given a triangulation T of a set of n points, Ash & Bolker [AB85] have shown that whether T is a Delaunay triangulation can be determined in O(n) time.

3. Graph Theoretic Properties of Proximity Graphs

Another area which has received little attention concerns the determination of graph theoretical properties of proximity graphs. The only proximity graphs which have been carefully examined are the Gabriel graph [MS80] and the RNG [Ur83].

4. Probabilistic Properties of Proximity Graphs

Yet another area which has received little attention concerns the determination of probabilistic and statistical properties of proximity graphs. The only proximity graphs which have been carefully examined are the Delaunay triangulation, the Gabriel graph, and the RNG. Miles [Mi70] has done considerable work on the probability distribution of random variables describing characteristics of the Delaunay triangulation. See also Getis & Boots [GB78]. Devroye [De88] obtains a variety of results concerning the expected number of edges in proximity graphs such as the Gabriel graph, the RNG and several types of nearest neighbour graphs. No results of this type are known polygonal approximation of curves see [Fi92] and [Ve92].

1.1.2 The Relative Neighborhood Graph

In [JK89] it is shown that the RNG in 3-space can be computed in $O(n^2 \log n)$ time and $O(\mu_3(S))$ space where $\mu_3(S)$ denotes the size of RNG(S). It is an open question whether this upper bound can be improved. It is also not known how large $\mu_3(S)$ can be over all instances of S. Denote this value by $\mu_3(n)$. It is shown in [JK89] that $\mu_3(n) = O(n^{(3/2)+c})$ where c is a positive constant and they conjecture that $\mu_3(n) = O(n)$.

1.1.3 β-Skeletons

In [KR85] it was shown that lune-based β-skeletons with $\beta > 1$ could be computed in $O(n^2)$ time. In [JKY89] it is shown that lune-based β-skeletons with $1 \leq \beta \leq 2$ can be constructed in linear time from the Delaunay triangulation in any L_p metric. The Delaunay triangulation in any L_p metric can be computed in O(n log n) time [Le80]. It is an open question whether for $\beta > 2$ these skeletons can be computed in $o(n^2)$ time.

1.1.4 The Sphere of Influence Graph

Avis and Horton [AH85] showed that the number of edges in the sphere-of-influence graph is bounded above by 29n. The best upper bound to date is 17.5. This follows from a lemma of Bateman in geometrical extrema suggested by a lemma of Besicovitch (*Geometry*, May 1951, pp. 667-675) and an observation of Kachalski. Bateman's lemma gives 18n and Kachalski's trick reduces it by.5. The same trick reduces Avis & Horton's bound by.5. David Avis conjectures that the best upper bound is 9n.

1.2 Polygon decomposition

1.2.1 Simple polygons

The problems of decomposing simple polygons into various types of more structured polygons have a number of practical applications and have received considerable attention recently from the theoretical perspective. See [To88a] for several papers discussing recent issues. In pattern recognition it is desired to obtain decompositions into meaningful parts. The so-called *component-directed* methods decompose the polygon into well established classes of simpler polygons such as convex or star-shaped polygons. These decompositions are satisfactory from the morphological point of view only rarely. Another approach which may be superior is to use *procedure-directed* methods based on proximity graphs. In [To80b] it was proposed to use the *relative-neighbour decomposition* (RND) of a simple polygon P of n vertices and an $O(n^3)$ time algorithm for its computation was given. ElGindy and Toussaint [ET88] reduced this complexity to $O(n^2)$. Two vertices p_i and p_j of a simple polygon are relative neighbours if their lune contains no other vertices of P that are visible from either p_i or p_j. Two vertices p_i and p_j are said to be visible if the line segment

Some Unsolved Problems on Proximity Graphs*

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ABSTRACT

Recent developments in the field of *computational morphology* (spatial and cluster analysis, computer vision, pattern recognition, computational perception, etc.) are making ever increasing use of proximity graphs. Thus it becomes increasingly relevant to understand the properties of such graphs as well to design efficient algorithms for their computation. In this note we mention some open problems in this area.

1. Computational Morphology

1.1 The Shape of a Set of Points

1.1.1 Introduction

One of the central problems in shape analysis is extracting the shape of a set of points. Let $S=\{x_1, x_2,..., x_n\}$ be a finite set of points in the plane. The relative neighborhood graph (RNG) [To80a] and the β-skeletons [KR85] are two structures that have been well investigated in this context. The RNG is obtained by joining two points x_i and x_j of S with an edge if Lune(x_i, x_j) does not contain any other points of S in its interior. Lune(x_i, x_j) is defined as the intersection of the two discs centered at x_i and x_j with radius equal to the distance between x_i and x_j. One of the best known proximity graphs on a set of points is the Delaunay triangulation (DT) and it is well known that the DT is a supergraph of the RNG [To80a]. The β-skeletons are a generalization of RNG's and Gabriel graphs and the lune-based neighborhoods in question are a function of a parameter β. In [To88b] a new graph termed the *sphere-of-influence* graph is proposed as a primal sketch intended to capture the low-level perceptual structure of visual scenes consisting of dot-patterns (point-sets). The graph suffers from none of the drawbacks of previous methods and for a dot pattern consisting of n dots can be computed efficiently in O(n log n) time. For a survey of the most recent results in this area see the paper by Radke [Ra88]. For applications of proximity graphs to

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