Computing the Constrained Euclidean, Geodesic and Link Centre of a Simple Polygon with Applications

Prosenjit Bose[*] Godfried Toussaint [†]

Abstract

Given an n vertex simple polygon M, we show how to compute the Euclidean center of M constrained to lie in the interior of M, in a polygonal region inside M or on the boundary of M in $O(n \log n + k)$ time where k is the number of intersections between M and the furthest point Voronoi diagram of the vertices of M. We show how to compute the geodesic center of M constrained to the boundary in $O(n \log n)$ time and the geodesic center of M constrained to lie in a polygonal region in $O(n(n + k))$ time where k is the number of intersections of the geodesic furthest point Voronoi diagram of M with the polygonal region. Furthermore, we show how to compute the link center of M constrained to the boundary of M in $O(n \log n)$ time. Finally, we show how to combine several of these criteria. For example, how to find the points whose maximum Euclidean and Link distance are minimized.

Computing such locations has applications in such diverse fields as Geographic Information Systems (G.I.S.) and the manufacturing industry. The following problem was one of the main motives of this work. In the manufacturing industry, finding a suitable location for the *pin gate* (the pin gate is the point from which liquid is poured or injected into a mold) is a difficult problem when viewed from the fluid dynamics of the molding process. However, experience has shown that a suitable pin gate location possesses several geometric characteristics, namely the distance from the pin gate to any point in the mold should be small and the number of turns on the path from a point in the mold to the pin gate should be small [19], [31]. The locations that possess these geometric characteristics are the constrained centres discussed above.

1 Introduction

In the manufacturing industry, two of the most popular classes of production methods are injection molding and gravity casting. Each of the two methods produces an object by filling a mold or cast of the given object with a liquid, and removing the object once the liquid has hardened (see Figure 1). The difference between the two methods is that liquid is injected into the mold in injection molding whereas liquid is poured into the mold and gravity is the sole force acting on the liquid in gravity casting [11]. A mold or cast, as defined in [5], refers to the whole assembly of parts that make up a cavity into which liquid is poured to give the shape of the desired component when the liquid hardens.

In this paper, we consider the problem of determining a suitable location for the *pin gate*. The *pin gate* is the point on the mold from which the liquid is poured or injected into the cavity. The location of the pin gate plays an important role in determining whether or not an object built by one of the two manufacturing processes will have surface defects. Many factors play a role in determining a suitable location for the pin gate when considered from the point of view of fluid dynamics and physics of the whole molding process. To date, trial and error, guided by engineering experience, has been the main method in determining a suitable location for the pin gate [19], [31], [40]. However, through this experience, a few of the key characteristics of an ideal location for a pin gate have been uncovered.

[*]Research supported by an NSERC and a FIR Grant. Département de Mathématiques et d'Informatique, Université du Québec à Trois-Rivières, C.P. 500, Trois-Rivieŕes, Québec G9A 5H7, Canada. E-mail: jit@uqtr.uquebec.ca.

[†]This work is supported in part by an NSERC and FCAR Grant. School of Computer Science, McGill University, Montréal, Québec, Canada, H3A 2A7. E-mail: godfried@mull.cs.mcgill.ca

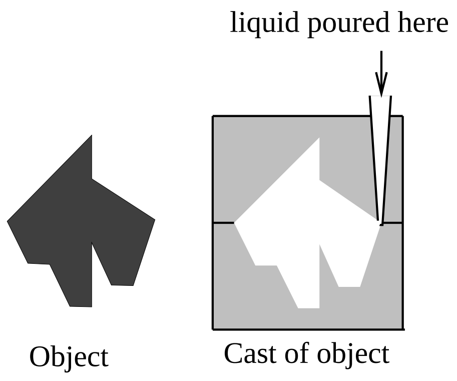


Figure 1: An object and its cast.

> If the distance from the gate to the extremities of the mold cavity is too great, the metal freezes prematurely, and misruns result. [19]

This quote points out one of the key problems faced by cast designers. In order to avoid this problem, designers must place the pin gate at a location where the distance from it to the extremities of the mold cavity is not too great. Another key characteristic of casts that leads to surface defects is the presence of many "sharp corners or overhanging or protruding sections..." [1]. These "sharp corners" disrupt the flow of molten liquid leading to surface defects. Therefore, the pin gate must be placed in a location such that the flow of molten liquid from the gate does not encounter too many sharp corners or make too many turns. For an overview of the many other factors causing defects in molds and casts, the reader is referred to [19], [1].

These observations allow one to deduce the following properties for a good location for a pin gate:

Property 1: The maximum distance from the pin gate to any point in the object should be small.

Property 2: The maximum number of turns the liquid takes on its path from the pin gate to any point in the object should be small.

When viewed from a purely geometric perspective, these problems can indeed be solved optimally. The geometric solutions provide an initial approximation that can aid in the search for a suitable location. In this paper, we solve the pin gate location problem for molds modelled as simple polygons which find applications in polymer molds. In practice, many 3-dimensional objects are almost flat so that in effect they can be considered as 2-dimensional. Therefore the 2-dimensional theory is more important than may appear at first glance, and sheds some light on the 3-dimensional problem.

The two properties that a pin gate should satisfy have several geometric interpretations. Property 1 can be interpreted as the point inside the simple polygon whose maximum distance to any point in the object is minimized. If distance is measured in the Euclidean metric, this point is referred to as the constrained Euclidean center. Sometimes a pin gate is constrained to lie on the boundary of the mold. In such a case, Property 1 can be interpreted as the point on the boundary of the simple polygon whose maximum distance to any point in the polygon is minimized with respect to all points on the boundary. This point is referred to as the *boundary-constrained* Euclidean center. On the other hand, distance can be measured by the geodesic metric, i.e., the minimum distance the liquid must travel inside the mold to reach a destination. In this case, Property 1 places the pin gate at the geodesic center, which by definition is constrained to lie inside the polygon, and the *boundary-constrained* geodesic center, respectively.

Property 2 can be interpreted as the link metric. The link metric measures the number of turns in a path between two points. For example, if two points can be joined by a line segment, then they are at link distance 1. The points inside a simple polygon, whose link distance to any other point in the polygon is minimized, are referred to as the link center. If the pin gate is constrained to the boundary, then it is referred to as the *boundary-constrained* link center.

2 Constrained Euclidean Center

In this section, we show how to find the point inside a simple polygon P as well as the point on ∂P whose maximum Euclidean distance to every point of P is minimized. These points are known as the *Euclidean center constrained to lie in the polygon*, and the *Euclidean center constrained to lie on the boundary of the polygon*, respectively.

We first review the problem of finding the Euclidean center. Given a set S of n points in the plane, the Euclidean center is the center of the smallest circle enclosing the points of S. This problem has a rich history. We summarize as in [30]. The search for an efficient algorithm seems to have begun in 1860 by Sylvester [38]. Later, Rademacher and Toeplitz [32] noted that the smallest enclosing circle is unique and is either the circumcircle of three points of the set or defined by a diametrical pair. This immediately gives an $O(n^4)$ algorithm. Elizinga and Hearn [12, 13] improved this to $O(n^2)$. Much work was done from an Operations Research perspective by viewing the problem as a minimax facility location problem, where the Euclidean center is the point whose greatest distance to any point of the set is minimized [17, 39, 20]. An $O(n \log n)$ time solution to this problem was proposed by Shamos and Hoey [34], but Bhattacharya and Toussaint [6] pointed out some errors in [34] and subsequently proposed an alternate $O(n \log n)$ time solution. Preparata [29] and Melville [25] also proposed an alternate $O(n \log n)$ time solution. However, no $\Omega(n \log n)$ time lower bound for the problem was known. A search for a resolution to this problem ensued, culminating in the discovery of an elegant $\Theta(n)$ time solution to the problem by Megiddo [23].

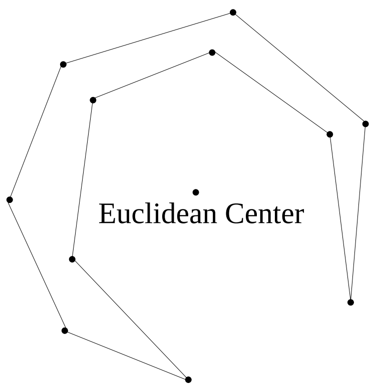


Figure 2: Euclidean center outside polygon.

The Euclidean center of the vertices of a simple polygon may be a good candidate for the location of the pin gate, but the center might lie outside the polygon (see Figure 2). Therefore, the location of the center must be constrained to lie inside the polygon or on its boundary since otherwise it cannot serve as a pin gate. Therefore, given an object modelled as a simple n vertex polygon, we wish to find the point lying inside the polygon whose maximum Euclidean distance to any point is minimized with respect to all points in the polygon. Since the furthest neighbor of a point must be a vertex, we can restrict our attention to finding the point lying inside the polygon whose maximum Euclidean distance to any vertex is minimized with respect to all points in the polygon. We also want the point on the boundary whose maximum Euclidean distance to any vertex is minimized with respect to all points on the boundary. Although the Euclidean center is unique, the Euclidean center constrained to lie inside the polygon as well as the Euclidean center constrained to lie on the boundary of the polygon need not be unique, as depicted in Figure 3.

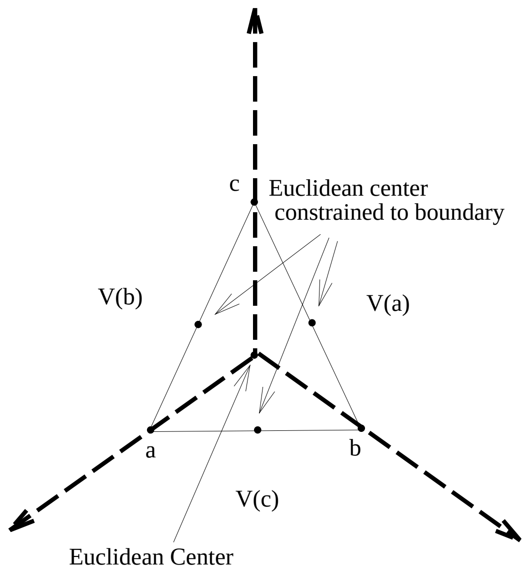


Figure 3: Constrained Euclidean center may not be unique.

2.1 Center Constrained to a Polygonal Region

We solve a slightly more general problem than the one mentioned in the introduction. Suppose we are given a set $S = \{s_1, s_2, \ldots, s_k\}$ of k points (in general position) in the plane E^2, and an n vertex simple polygon P. We wish to find the point c in P whose maximum distance to any point in S is minimized. If c is not constrained to lie in P, then it is the Euclidean center of S. However, we refer to c as the Euclidean center of S constrained to P and denote it by $EC_P(S)$.

Our algorithms make use of the *furthest point Voronoi diagram* of the set S, denoted as $FPVD(S)$. Given a point $x \in E^2$, we let $\phi(x)$ denote the furthest neighbors of x in S, that is the set of points in S such that $d(x, \phi(x)) = \max_{y \in S} d(x, y)$ where d is the Euclidean distance function. The $FPVD(S)$ partitions the plane into unbounded convex cells, $V(s_i)$, such that for any point $p \in V(s_i)$, $s_i \in \phi(p)$. This structure can be computed in $O(n \log n)$ time [30]. A list of the many geometric properties of the furthest point Voronoi diagram can be found in [26, 30, 6].

We first review some properties of the Euclidean center which will help us in finding its constrained counterpart.

Lemma 2.1 *[30, 6] The Euclidean center of S lies on the midpoint of the diameter of the set S, $DIAM(S)$, provided that the circle with $DIAM(S)$ as diameter contains the set S.*

Lemma 2.2 *[30, 6] If the Euclidean center does not lie on the midpoint of $DIAM(S)$, then it lies on the vertex of the $FPVD(S)$ that yields the smallest spanning circle.*

These two lemmas characterize the location of the Euclidean center. When considering the constrained version of the problem, notice that if the Euclidean center happens to lie inside the constraining polygon, then it is also the constrained Euclidean center. However, difficulties arise when the Euclidean center does not lie inside the polygon. These difficulties are resolved in the following lemmas.

Lemma 2.3 *The Euclidean center of S constrained to lie in P is the midpoint of $DIAM(S)$ provided that the diametral circle contains the set S, and the midpoint is contained in P.*

Proof: Follows from Lemma 2.1. ■

Before tackling the problem of determining the location of $EC_P(S)$ when it is not on the midpoint of $DIAM(S)$, we first establish a lemma that will prove useful. Let a, b be two points in S such that both a and b are on the convex hull of S, $[ab]$ is not the diameter of S, and $V(a)$ and $V(b)$, the two cells of $FPVD(S)$ representing a and b, respectively, are adjacent and separated by an edge e. Let x be a point on the interior of e, and let $\epsilon > 0$ be any small constant.

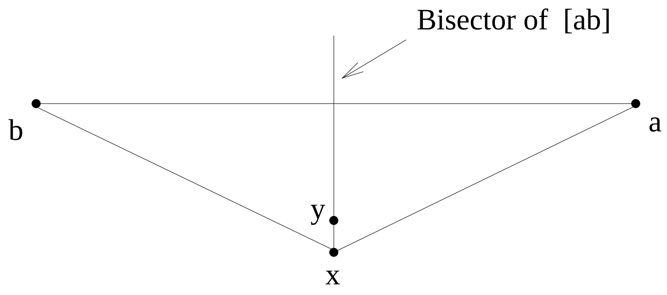


Figure 4: Illustration for proof of Lemma 2.4.

Lemma 2.4 *There exists a point $y \in e$ with $d(x, y) < \epsilon$ such that $d(y, a) < d(x, a)$ and $d(y, b) < d(x, b)$.*

Proof: See Figure 4. The edge e must lie on the bisector of line segment $[ab]$, since the points on e are equidistant from both a and b. The points a, b, x must form a triangle because otherwise $[ab]$ would be the diameter. Since x is contained in $int(e)$, let y be a point on e in $\triangle(abx)$ such that $d(x, y) < \epsilon$. The lemma follows. ■

Lemma 2.5 *If $\|S\| > 1$ then a point b of S cannot lie in $V(b)$.*

Proof: Let $x \in S$ be a point distinct from b. Note that $d(b, b) = 0$, however, $d(b, x) > 0$ which contradicts the fact that $b \in V(b)$. ■

We now complete the characterization of $EC_P(S)$.

Lemma 2.6 *If the Euclidean center of S constrained to lie in P is not the midpoint of $DIAM(S)$, then it lies on one of the following points that yields the smallest spanning circle:*

1. *a vertex of the $FPVD(S)$ contained in P,*
2. *a proper intersection point of the $FPVD(S)$ and the boundary P,*
3. *a vertex of the polygon P,*
4. *a point x on an edge e of P with the property that $\forall y \in e$, if $\phi(y) = \phi(x)$ then $d(y, \phi(x)) \geq d(x, \phi(x))$.*

Proof: If $EC_P(S)$ does not lie on any of the points mentioned in the statement of the lemma, then it must lie in one of the regions described in the following four cases. We show that each of these cases leads to a contradiction. For simplicity of exposition, let $c = EC_P(S)$.

Case 1: c is a point in the interior of a cell of the $FPVD(S)$, and in $int(P)$. Let $V(b)$ be the cell containing c. By the Jordon Curve Theorem [27], line segment $[bc]$ must intersect ∂P or $V(b)$ since $b \notin V(b)$ by Lemma 2.5. Let x be the intersection point closest to c. The point x must be in $V(b)$. Therefore the circle centered at x with radius $d(x, b)$ encloses the set S. However, $d(x, b) < d(c, b)$ by construction. Hence, we have a contradiction.

Case 2: c is a point in the interior of a cell of the $FPVD(S)$, and in the interior of an edge e of P but does not satisfy the property that $\forall y \in e$, if $\phi(y) = \phi(c)$ then $d(y, \phi(c)) \geq d(c, \phi(c))$. Since the latter property is not satisfied, a point $x \in e$ such that $\phi(x) = \phi(c)$ and $d(x, \phi(c)) < d(c, \phi(c))$ must exist. However, the very existence of x contradicts that c is the constrained Euclidean center since the circle centered at x with radius $d(x, \phi(c))$ encloses S.

Case 3: c is a point in the interior of an edge e of the $FPVD(S)$, and in $int(P)$. Let $V(a)$ and $V(b)$ be the two cells separated by the edge e. Since c is not on the diameter of the set S, by Lemma 2.4 we know that there exists a point x in e and in $int(P)$ such that $d(x,a) < d(c,a)$ and $d(x,b) < d(c,b)$. This contradicts that c is the constrained Euclidean center.

Case 4: c is a point in the interior of an edge e_v of the $FPVD(S)$, and in the interior of an edge e_p of P such that e_v and e_p intersect but not properly. Same argument as Case 3.

■

Lemma 2.3 and Lemma 2.6 characterize the location of $EC_P(S)$. We outline the following algorithm to compute this point.

Algorithm 1: *Euclidean Center of P constrained to lie in S*

Input: A set of points $S = \{s_1, s_2, \ldots, s_n\}$ and a simple polygon $P = \{p_1, p_2, \ldots, p_n\}$.

Output: $EC_P(S)$

1. Compute the $FPVD(S)$.
2. Compute $DIAM(S)$.
3. Compute the circle C having $DIAM(S)$ as diameter.
4. Preprocess P in $O(n \log n)$ time for point inclusion testing in $O(\log n)$ time using the algorithms of Kirkpatrick [18] or Sarnak and Tarjan [33].
5. If the midpoint of C is contained in P and all the points of S are contained in C then exit with the midpoint of $DIAM(S)$.
6. Compute the set of vertices of $FPVD(S)$ contained in P. Let V_c represent this set.
7. Compute the set of intersections $I_c = \{i_1, i_2, \ldots, i_k\}$ of P with $FPVD(S)$.
8. Partition each edge e_i of P such that for every pair of points $x, y \in e_i$, we have that $\phi(x) = \phi(y)$. Denote the j^{th} partition of e_i by e_{ij}.
9. For each e_{ij}, compute the point on e_{ij} closest to $\phi(e_{ij})$. If this point is not an endpoint of e_{ij}, place it in the set E_c.
10. Let P_c represent the vertices of P. For each point c in V_c, I_c, P_c, E_c, compute the smallest spanning circle with center c. Let SP represent this set.
11. Select all the smallest circles in SP, and output their centers and the radius.

Notice that we assumed that the number of vertices of P equals the number of points in S. Clearly, this need not be the case, however, this assumption simplifies the complexity of notation. It is straightforward to repeat the complexity analysis when P and S have different cardinalities.

Theorem 2.1 *Given a set of points $S = \{s_1, s_2, \ldots, s_n\}$ and a simple polygon $P = \{p_1, p_2, \ldots, p_n\}$, we can compute the Euclidean center of S constrained to lie in P in time $O(n \log n + k)$ where n is the size of the input and k is the number of intersections between the edges of the $FPVD(S)$ and P.*

Proof: The correctness of the algorithm follows from Lemmas 2.3 and 2.6.

Let us analyze the complexity of the algorithm. Step 1 of the algorithm can be computed in $O(n \log n)$ time using the algorithm of Shamos [30]. Step 2 can be computed in $O(n \log n)$ time by first computing the convex hull of S and then finding the diameter of the convex hull. Preprocessing for point inclusion can be done in $O(n \log n)$ using the algorithm of Kirkpatrick [18] or Sarnak and Tarjan [33]. Step 5 can be achieved in $O(n \log n)$ time by using the point inclusion test. Step 6 can be done in $O(n \log n)$ time using the point inclusion test. Step 7 can be computed in $O(n \log n + k)$ time where k is the number of intersections between P and $FPVD(S)$ using the algorithm of Chan [7]. If we color the segments in $FPVD(S)$ blue and the edges of P red, then the algorithm of [7] reports the intersections along each edge of P in sorted order. Once these intersection points have been computed, Step 8 and 9 can be achieved in $O(n + k)$ time. Step 10 can be computed in $O(n + k)$ time since it takes constant time to compute the circle and there are $O(n + k)$ points in the set SP. Finally, Step 11 can be computed in $O(n + k)$. Therefore, the total complexity of the algorithm is $O(n \log n + k)$ time. ■

For simple polygons, k can be $O(n^2)$, however, for convex polygons, we notice the following: a line segment can intersect a convex polygon only twice. Therefore, since $FPVD(S)$ consists of $O(n)$ line segments, there can only be $O(n)$ intersections between $FPVD(S)$ and an n vertex convex polygon. Therefore, we have:

Corollary 2.1 *Given a set of points $S = \{s_1, s_2, \ldots, s_n\}$ and a convex polygon $P = \{p_1, p_2, \ldots, p_n\}$, we can compute the Euclidean center of S constrained to lie in P in time $O(n \log n)$ where n is the size of the input.*

2.2 Center Constrained to a Polygonal Chain

With a slight modification, Algorithm 1 can compute the Euclidean center constrained to lie on the boundary of the polygon, denoted as $EC_{\partial P}(S)$. These modifications are outlined below.

Lemma 2.7 *The Euclidean center of S constrained to lie on the boundary of P is the midpoint of $DIAM(S)$ provided that the diametral circle contains the set S, and the midpoint is on the boundary of P.*

Proof: Follows from Lemma 2.1. ■

Lemma 2.8 *If the Euclidean center of S constrained to lie on the boundary of P is not the midpoint of $DIAM(S)$, then it lies on one of the following points that yields the smallest spanning circle:*

1. *a vertex of the $FPVD(S)$ on the boundary of P,*
2. *an intersection point of the $FPVD(S)$ and the boundary P,*
3. *a vertex of the polygon P,*
4. *a point x on an edge e of P with the property that $\forall y \in e$, if $\phi(y) = \phi(x)$ then $d(y, \phi(x)) \geq d(x, \phi(x))$.*

Proof: If $EC_{\partial P}(S)$ does not lie on any of the points mentioned in the statement of the lemma, then it must lie in one of the regions described in the following four cases. We show that each of these cases leads to a contradiction. For simplicity of exposition, let $c = EC_{\partial P}(S)$.

Case 1: c is a point in the interior of a cell of the $FPVD(S)$, and in $int(P)$. This cannot happen since c must be on the boundary of P.

Case 2: c is a point in the interior of a cell of the $FPVD(S)$, and in the interior of an edge e of P but does not satisfy the property that $\forall y \in e$, if $\phi(y) = \phi(c)$ then $d(y, \phi(c)) \geq d(c, \phi(c))$. Since the latter property is not satisfied, a point $x \in e$ such that $\phi(x) = \phi(c)$ and $d(x, \phi(c)) < d(c, \phi(c))$ must exist. However, the very existence of x contradicts that c is the constrained Euclidean center since the circle centered at x with radius $d(x, \phi(c))$ encloses S.

Case 3: c is a point in the interior of an edge of the $FPVD(S)$, and in $int(P)$. Again, c cannot lie in $int(P)$ since it is constrained to the boundary.

Case 4: c is a point in the interior of an edge e_v of the $FPVD(S)$, and in the interior of an edge e_p of P such that e_v and e_p intersect but not properly. Let $V(a)$ and $V(b)$ be the two cells separated by the edge e_v. Since c is not on the diameter of the set S, by Lemma 2.4 we know that there exists a point x in e_v and in e_p such that $d(x, a) < d(c, a)$ and $d(x, b) < d(c, b)$. This contradicts that c is the constrained Euclidean center.

■

Lemma 2.7 and Lemma 2.8 characterize the location of $EC_{\partial P}(S)$. The modifications to Algorithm 1 for computing these points are straightforward. Therefore, we conclude with the following.

Theorem 2.2 *Given a set of points $S = \{s_1, s_2, \ldots, s_n\}$ and a simple polygon $P = \{p_1, p_2, \ldots, p_n\}$, we can compute the Euclidean center of S constrained to lie on the boundary of P in time $O(n \log n + k)$ where n is the size of the input and k is the number of intersections between the edges of the $FPVD(S)$ and P.*

Corollary 2.2 *Given a set of points $S = \{s_1, s_2, \ldots, s_n\}$ and a convex polygon $P = \{p_1, p_2, \ldots, p_n\}$, we can compute the Euclidean center of S constrained to lie on the boundary P in time $O(n \log n)$ where n is the size of the input.*

3 Constrained Geodesic Center

Both versions of the constrained Euclidean center serve as good first approximations for the locations of the pin gate. However, in some cases the constrained Euclidean center may not be a point satisfying Property 1, as intended (see Figure 5). In fact, it may be quite bad in the sense that the liquid may have to travel quite far despite the fact that the pin gate is located at the constrained or boundary-constrained Euclidean center. The reason is that the Euclidean distance of the pin gate to all the points may not be a good measure of the actual distance the liquid must travel inside the polygon. For example, in Figure 5, the Euclidean center, constrained Euclidean center and boundary-constrained Euclidean center all lie on the same vertex indicated on the polygon. However, the distance that the liquid must travel inside the polygon from that point to vertex v is quite large compared to vertex c. Although for convex or "near" convex objects, the Euclidean metric may be good, it seems that the geodesic metric may serve as a better approximation since liquid is travelling inside the polygon.

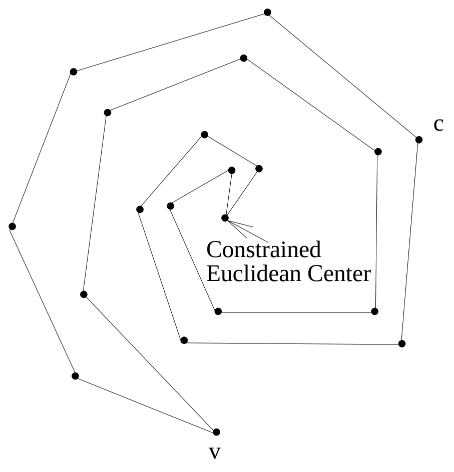


Figure 5: Constrained Euclidean center may not be a good approximation for the best pin gate location.

In the geodesic metric, the distance between two points inside a simple polygon is defined as the length of the shortest path connecting the two points inside the polygon. The *geodesic center* of a simple polygon is the point whose maximum geodesic distance to any other point in the polygon is minimized. Therefore, by definition, the geodesic center of a simple polygon lies inside the polygon. Although the geodesic center and boundary-constrained geodesic center may serve as better approximations for the location of a pin gate, computing both centers is more difficult than their Euclidean counter-parts as we shall see.

3.1 Geometric Properties

The problem of computing the geodesic center of a simple n vertex polygon P, denoted $GC(P)$, was first tackled by Asano and Toussaint [4]. They gave an $O(n^3 \log\log n)$ time algorithm for computing the center. In [4], it is shown that the geodesic center is unique and located on a vertex of the *geodesic furthest point Voronoi diagram* of P, denoted $GFPVD(P)$. The $GFPVD(P)$, like its Euclidean counter-part, divides the polygon P into cells $V(v_i)$, such that the locus of points in $V(v_i)$ is further from v_i than any other vertex of P (with distance measured with the geodesic metric). Later, Pollack, Rote and Sharir [28] reduced the complexity of computing the geodesic center to $O(n \log n)$ time. They used a different approach and achieved their time bound by a modification of Meggido's technique. Recently, Aronov et al.[2] presented an $O(n \log n)$ time algorithm for computing the $GFPVD(P)$, thus providing an alternate $O(n \log n)$ time solution for computing the center. Therefore, to compute the geodesic center of a simple polygon, any one of the above algorithms may be used, however, all of these algorithms are complicated and involved.

The problem of computing the boundary-constrained geodesic center of a simple polygon P, denoted as $GC(\partial P)$, has not previously been addressed. We concentrate on solving this problem. Like its Euclidean counter-part, the geodesic center constrained to the boundary is not necessarily unique, and not necessarily an intersection point of $GFPVD(P)$ and P. Figure 3 shows an example of this. If an algorithm for computing the geodesic center already exists, the following heuristic may serve as a good approximation of the boundary-constrained geodesic center.

Heuristic 3.1 *A heuristic for computing the boundary-constrained geodesic center is to compute the point on the boundary closest to the geodesic center.*

In some cases, this heuristic actually gives the boundary-constrained geodesic center, as seen in Figure 3. In the next section, we present an $O(n \log n)$ time algorithm to compute the boundary-constrained geodesic center exactly. The main idea behind the algorithm is the following. We divide the polygon boundary into polygonal chains such that the geodesic furthest neighbor of any point on a given chain is the same. Then, we compute for each chain, the point, which we call the candidate for that chain, whose distance to the furthest neighbor is the smallest compared to any other point on the chain. We select the smallest candidates as the geodesic center constrained to the boundary. We modify an algorithm of Suri [36], similar to [2], to compute this.

Given two points a, b in a polygon P, there is a unique geodesic path connecting a, b in P. We denote this path by $\pi(a, b)$ and its length by $d_G(a, b)$. Since geodesic distance is a metric, the triangle inequality holds. Therefore, we have that $d_G(x, y) \leq d_G(x, z) + d_G(z, y)$ for every three points x, y, z in P. The *geodesic furthest neighbors* of a point x in P, denoted by $\phi(x)$, are the set of points y in P such that $d_G(x, y) = \max_{\forall\ z \in P}\{d_G(x, z)\}$. Asano and Toussaint [4] showed that the geodesic furthest neighbor of a point is always a convex vertex of the polygon. The *geodesic diameter* of a polygon P, denoted as $GDIAM(P)$, is determined by the pair of points in P whose geodesic distance is maximum over all pairs of points in P. If two shortest paths do not share a point, we say they are *disjoint*; otherwise, we say that the paths *intersect*.

An important property of geodesics, at the heart of the algorithm, is the *Crossing Property* described in the following lemma.

Lemma 3.1 (Crossing Property) *[36] Let p_1, p_2, p_3, p_4 be four points in this order on the boundary of P. Suppose that $p_3 \in \phi(p_2)$ and $p_4 \in \phi(p_1)$. Then we also have $p_3 \in \phi(p_1)$ and $p_4 \in \phi(p_2)$.*

To compute the boundary-constrained geodesic center, we first compute a constrained geodesic decomposition of the boundary of polygon P, which is a decomposition of the boundary of P into polygonal chains $(c_1, c_2, \ldots, c_t)$ such that $\bigcup_{i=1}^{t} c_i = P$ and for every $x, y \in c_i$, $\phi(x) = \phi(y)$. We denote this decomposition as ∂-$CGD(P)$. Given this decomposition, the constrained geodesic center can be easily identified, as shall be shown in the next section. The crossing property is the key behind the algorithm. It suggests a divide-and-conquer approach to solving the problem of computing the constrained geodesic decomposition of ∂P. We first consider a restricted version of the decomposition problem, and then we show how to use its solution to compute the whole decomposition.

3.2 Restricted Geodesic Decomposition

The restricted version of the decomposition problem is described as follows. Let $U = (u_a, \ldots, u_b)$ be the counterclockwise chain from point u_a to point u_b on the boundary of P. Let $V = (v_c, \ldots, v_d)$ be the clockwise chain from v_c to v_d on the boundary of P, such that both chains are disjoint except possibly at the endpoints. The set of points which are the furthest neighbors of x restricted to V is denoted by $\phi_V(x)$. We want to decompose U into polygonal chains $(c_1', c_2', \ldots, c_s')$ such that $\bigcup_{i=1}^{s} c_i' = U$ and for every $x, y \in c_i'$, $\phi_V(x) = \phi_V(y)$. We refer to this decomposition as the restricted decomposition of U with respect to V, denoted by $RGD_V(U)$.

Let u_a, u_b, v_d, v_c be four points on ∂P appearing in that order in a counterclockwise traversal of ∂P. $P[u_a, u_b; v_c, v_d]$ denotes the region of P obtained by joining the counterclockwise chain of ∂P from u_a to u_b and the clockwise chain of ∂P from v_c to v_d with $\pi(u_a, v_c)$ and $\pi(u_b, v_d)$ (see Figure 6). We say that a polygonal region $R \subset P$ is *geodesically convex* if for every pair of points $x, y \in R$, we have that $\pi(x, y) \in R$.

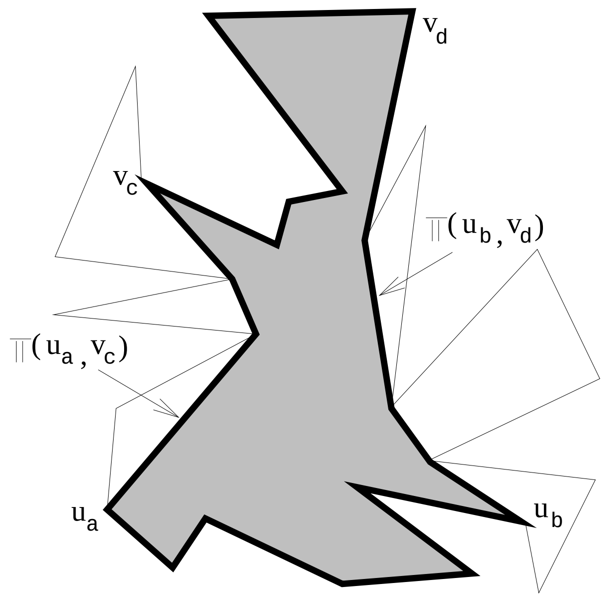


Figure 6: $P[u_a, u_b; v_c, v_d]$ is shaded.

Lemma 3.2 *[36] $P[u_a, u_b; v_c, v_d]$ is geodesically convex.*

Lemma 3.2 implies that the restricted geodesic decomposition of U with respect to chain V can be done entirely within $P[u_a, u_b; v_c, v_d]$. Below, we outline the algorithm to compute this decomposition.

Algorithm 2: $RGD(P[u_a, u_b; v_c, v_d])$

```
1. If $P[u_a, u_b; v_c, v_d]$ is degenerate then
       Find a $v_m \in [v_c, \ldots, v_d]$ such that $d_G(u_a, v_m) = \max\{d_G(u_a, v_j) | c \le j \le d\}$;
```

```
        Find the point $u_{\min} \in [u_a, \ldots, u_b]$ such that $d_G(u_{\min}, v_m)$
        $= \min\{d_G(u_i, v_m) | a \leq i \leq b\}$.
        Output $u_{\min}$, $v_m$, $[u_a, \ldots, u_b]$, $d_G(u_{\min}, v_m)$.
  2. Else if $(b - a) \leq 2$ and $u_a, u_b$ are vertices then
        Determine $\phi(u_a)$ and $\phi(u_b)$.
        If $\phi(u_a) = \phi(u_b)$ then
          let $v_m = \phi(u_a)$.
          Find point $u_{\min}$ on $[u_a, u_b]$ such that $d_G(u_{\min}, v_m)$
          $= \min\{d_G(p, v_m) | p \in [u_a, u_b]\}$.
          Output $u_{\min}$, $v_m$, $[u_a, u_b], d_G(u_{\min}, v_m)$.
        Else
          Compute partition points $x_1, x_2, \ldots, x_s$ on edge $[u_a u_b]$.
          Sort these partition points including the endpoints.
          Let $[u_1, u_2, \ldots, u_{s+2}]$ be the points ordered on edge $[u_a, u_b]$.
          let $k = \lceil (s + 2)/2 \rceil$.
          Find a $v_m \in [v_c, \ldots, v_d]$ such that $d_G(u_k, v_m)$
          $= \max\{d_G(u_k, v_j) | c \leq j \leq d\}$
          Construct and triangulate $P[u_a, u_k; v_m, v_d]$ and $P[u_k, u_b; v_c, m]$;
          Call *RGD*($P[u_a, u_k; v_m, v_d]$) and *RGD*($P[u_k, u_b; v_c, v_m]$).
  3. Else if $(b - a) \leq 2$ and $u_a, u_b$ are not both vertices then
        Determine $\phi(u_a)$ and $\phi(u_b)$.
        If $\phi(u_a) = \phi(u_b)$ then
          let $v_m = \phi(u_a)$.
          Find point $u_{\min}$ on $[u_a, u_b]$ such that $d_G(u_{\min}, v_m)$
          $= \min\{d_G(p, v_m) | p \in [u_a, u_b]\}$.
          Output $u_{\min}$, $v_m$, $[u_a, u_b], d_G(u_{\min}, v_m)$.
        Else
          solve directly by computing upper envelopes.
  4. Else
        let $k = \lceil (a + b)/2 \rceil$
        Find a $v_m \in [v_c, \ldots, v_d]$ such that $d_G(u_k, v_m) = \max\{d_G(u_k, v_j) | c \leq j \leq d\}$
        Construct and triangulate $P[u_a, u_k; v_m, v_d]$ and $P[u_k, u_b; v_c, v_m]$;
        Call *RGD*($P[u_a, u_k; v_m, v_d]$) and *RGD*($P[u_k, u_b; v_c, v_m]$).
```

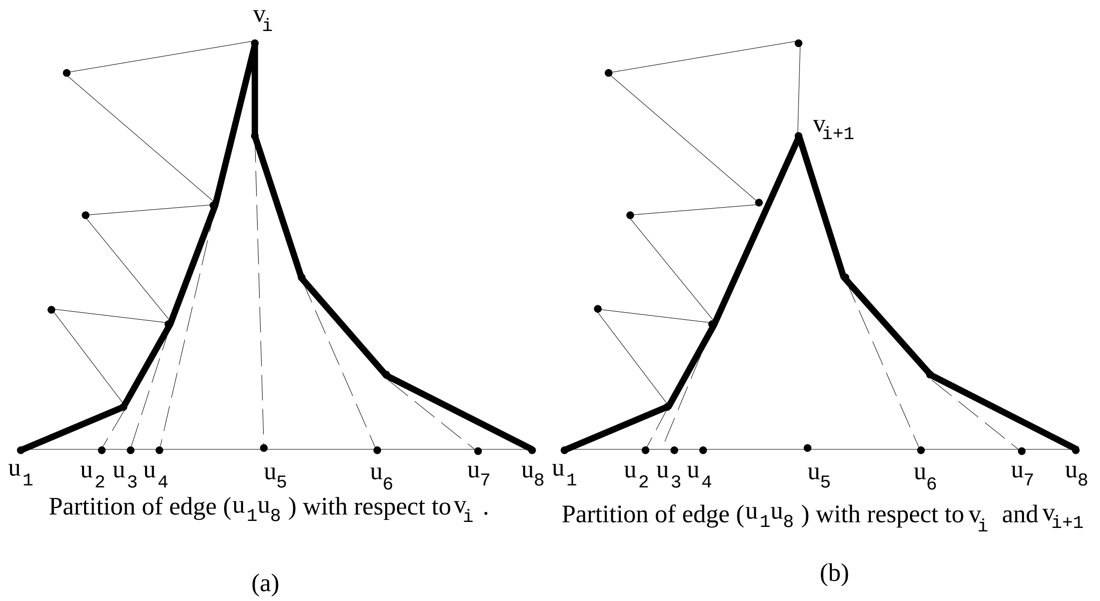
Not only does algorithm *RGD* compute the restricted geodesic decomposition of U into polygonal chains $(c_1{}', c_2{}', \ldots, c_s{}')$ such that $\bigcup_{i=1}^{s} c_i{}' = U$ and for every $x, y \in c_i{}'$, $\phi_V(x) = \phi_V(y)$, but for each chain $c_i{}'$ it computes the point on the chain whose distance to its furthest neighbor is minimum. We prove the correctness of the algorithm and at the same time, elaborate on some of the steps such as how to compute the partition in Step 2.

We say that $P[u_a, u_b; v_c, v_d]$ is *degenerate* if $\pi(u_a, v_c)$ and $\pi(u_b, v_d)$ are not disjoint. Given a degenerate instance of $P[u_a, u_b; v_c, v_d]$ computing the decomposition of U with respect to V is straightforward. Let x be a point on $\pi(u_a, v_c) \cap \pi(u_b, v_d)$. Every shortest path between a point y in U and a point z in V must contain x. Therefore, the point v_f of V furthest from x is also the point furthest from all points in U. The point v_f can be computed by traversing the shortest path tree of x. Since this tree can be computed in linear time [16], we conclude with the following.

Lemma 3.3 *If $P[u_a, u_b; v_c, v_d]$ is degenerate and v_f is the furthest point from $x \in \pi(u_a, v_c) \cap \pi(u_b, v_d)$, then the point v_f on V is $\phi_V(z)$ for all $z \in U$. The point v_f can be computed in time proportional to the size of $P[u_a, u_b; v_c, v_d]$.*

Given an instance of $P[u_a, u_b; v_c, v_d]$, if $(u_a, \ldots, u_b)$ is a polygonal chain, then by the crossing property, we can divide the chain in half and recurse. However, if u_a and u_b are the endpoints of an edge, it is not clear how to proceed. In such a situation, we resolve the problem by partitioning the edge into subedges. We require the following property on each subedge s_i of $[u_a, u_b]$. For every pair of points x, y on s_i, we want the shortest path from x to every vertex v_j on $(v_c, \ldots, v_d)$ to be identical to the shortest path from y to v_j except for the first link. We refer to this property as the *path-invariant* property of a subedge.

To see how to compute a partition of the edge respecting the path-invariant property, let us look at Figure 7. In Figure 7(a), notice that once the shortest paths from v_i to u_1, and v_i to u_8 are computed, the path-invariant partition of the edge falls out by extending the edges in both paths to $[u_1, u_8]$. In this partition of the edge, every point on a subedge (u_i, u_{i+1}) has the same shortest path to v_i except for the first link. To extend this partition to two vertices, see Figure 7(b). In Figure 7(b), the partition with respect to v_i and v_{i+1} differs by only one point located between u_2 and u_3. Again, each subedge has the property that for every point on the subedge, the shortest paths to both v_i and v_{i+1} are the same except for the first link. By continuing in this manner, the edge can be partitioned with respect to the chain $(v_c, \ldots, v_d)$.



v_i
v_{i+1}
u_1 u_2 u_3 u_4 u_5 u_6 u_7 u_8
u_1 u_2 u_3 u_4 u_5 u_6 u_7 u_8
Partition of edge (u_1u_8) with respect to v_i .
Partition of edge (u_1u_8) with respect to v_i and v_{i+1} .
(a)
(b)

Figure 7: Partitioning an edge.

Let m be the size of $P[u_a, u_b; v_c, v_d]$. Since $P[u_a, u_b; v_c, v_d]$ is triangulated, the shortest path tree of u_a and u_b can be obtained in $O(m)$ time using the algorithm of [16]. Once the shortest path trees have been computed, all the partition points on the edge can also be obtained in O(m) time, by traversing the two trees. Finally, $O(m \log m)$ time is used to sort the partition points. Hence, we conclude with the following.

Lemma 3.4 *Given an instance of $P[u_a, u_b; v_c, v_d]$ of size m, where $[u_a, u_b]$ is an edge, we can partition in $O(m \log m)$ time the edge $[u_a, u_b]$ into subedges such that each subedge respects the path-invariant property with respect to the chain $(v_c, \ldots, v_d)$.*

The reason we partition the edge into subedges, when faced with an instance of $P[u_a, u_b; v_c, v_d]$ where $[u_a, u_b]$ is an edge, is quite simple. First, it allows us to continue the divide-and-conquer algorithm. Second, the base problem that we are faced with at the end of the recursion can be solved directly because of the path-invariant property. As the algorithm computes the decomposition of the U chain, eventually in Step 3, we are faced with an instance $P[u_a, u_b; v_c, v_d]$ where u_a, u_b are the endpoints of a subedge respecting the path-invariant property, and $(v_c, \ldots, v_d)$ is a polygonal chain. Because of the path-invariant property, we know that the distance from a vertex $v_i \in (v_c, \ldots, v_d)$ to a point x on $[u_a, u_b]$ has the form $k_1 + \sqrt{k^2 + z^2}$ where k_1 is a constant whose value is the geodesic distance from v_i to the last vertex, say v_l, before x on $\pi(v_i, x)$, and $\sqrt{k^2 + z^2}$ is the distance from v_l to x with k as the orthogonal distance between the line L containing $[u_a, u_b]$ and v_l, and z represents the distance between x and the point on L that is the orthogonal projection of v_l onto L. Consider the example in Figure 8. The constant k_1 accounts for the distance from v_i to v_{i+2}. By the path-invariant property, this value is the same for all points on the subedge. The distance from v_{i+2} to x is accounted for by $\sqrt{k^2 + z^2}$.

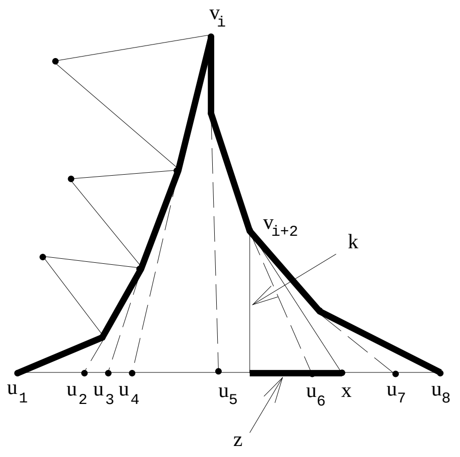


Figure 8: Distance function from subedge to vertex.

Let $d_{v_i}(x)$ denote the distance function from $v_i \in (v_c, \ldots, v_d)$ to a point x in $[u_a, u_b]$. These functions are simple and can be used to solve directly the decomposition of $[u_a, u_b]$ into subedges such that for each point in the subedge, the furthest neighbor is the same vertex of $(v_c, \ldots, v_d)$. This can be achieved by computing the upper envelope of the functions $d_{v_i}(x)$ for all $v_i \in (v_c, \ldots, v_d)$. The following lemma gives the key to solving this in time that is linear in the size of the problem instance.

Lemma 3.5 *Let p_1, p_2, p_3, p_4 be four points in this order on the boundary of P. If $d_G(p_1, p_4) > d_G(p_1, p_3)$ then $d_G(p_2, p_4) > d_G(p_2, p_3)$.*

Proof: Suppose $d_G(p_1, p_4) > d_G(p_1, p_3)$ and $d_G(p_2, p_3) \geq d_G(p_2, p_4)$. We see that $d_G(p_1, p_4) + d_G(p_2, p_3) > d_G(p_1, p_3) + d_G(p_2, p_4)$. By the relative positions of the points, $\pi(p_1, p_3)$ must intersect $\pi(p_2, p_4)$. Let x be a point on this intersection.

By the triangle inequality, $d_G(p_1, p_4) \leq d_G(p_1, x) + d_G(x, p_4)$, and $d_G(p_2, p_3) \leq d_G(p_2, x) + d_G(x, p_3)$. But since $x \in \pi(p_1, p_3) \cap \pi(p_2, p_4)$, we contradict our assumption, proving the lemma. ■

The above lemma implies that we can compute the upper envelope in linear time simply by inserting the functions in the order $(v_c, \ldots, v_d)$ or the reverse order. Both arguments are symmetric. Let us look at an example to see why this is so. Suppose we are inserting the functions in the order $d_{v_d}(x), d_{v_{d-1}}(x), \ldots, d_{v_c}(x)$. Consider the example in Figure 9 where the first three functions have been inserted. The upper envelope consists of $d_{v_d}(x)$ between u_a and u_1, $d_{v_{d-1}}(x)$ between u_1 and u_2, and $d_{v_{d-2}}(x)$ between u_2 and u_b. The next function to be added is $d_{v_{d-3}}(x)$. If $d_{v_{d-3}}(x)$ is below $d_{v_{d-2}}(x)$ between u_2 and u_b then it cannot lie on the upper envelope because if it did, we would have a situation contradicting Lemma 3.5. If $d_{v_{d-3}}(x)$ intersects $d_{v_{d-2}}(x)$ between u_2 and u_b then we update the upper envelope by adding the intersection point, but we no longer need to compare $d_{v_{d-3}}(x)$

with any other function on the upper envelope by Lemma 3.5. Finally, if $d_{v_{d-3}}(x)$ is above $d_{v_{d-2}}(x)$ between u_2 and u_b then remove the intersection point u_2, remove $d_{v_{d-2}}(x)$ from the upper envelope, and repeat the test on the next piece of the upper envelope, namely $d_{v_{d-1}}(x)$. Therefore, when we add the functions in the order $d_{v_d}(x), d_{v_{d-1}}(x), \ldots, d_{v_c}(x)$, the amount of time spent adding a function can be determined in constant time plus the time proportional to the number of functions and intersection points deleted which is linear time overall. We conclude with the following lemma.

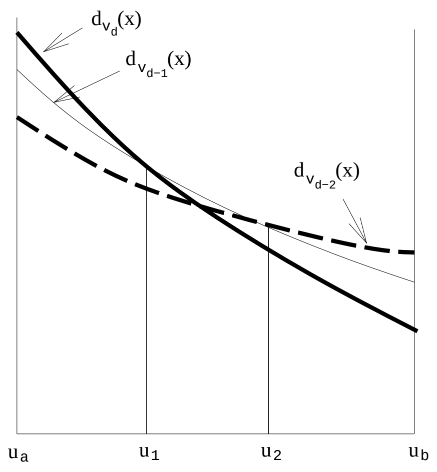


Figure 9: Computing upper envelopes.

Lemma 3.6 *Given an instance of $P[u_a, u_b; v_c, v_d]$ where $[u_a, u_b]$ is a segment with the path-invariant property, $[u_a, u_b]$ can be decomposed, in time proportional to the size of $P[u_a, u_b; v_c, v_d]$, into subedges such that for each point in the subedge, the furthest neighbor is the same vertex of $(v_c, \ldots, v_d)$.*

The algorithm to compute $RGD_V(U)$ stems from the crossing property described in Lemma 3.1. We show that this property holds at all levels of recursion. The algorithm is initiated with a call to $RGD(P[u_a, u_b; v_c, v_d])$. At each invocation, the algorithm either makes two recursive calls with smaller problem instances or solves the problem directly. The calling relation forms a binary tree, which we refer to as the *recursion tree*. A node of this tree having two children is an instance of *RGD* where two recursive calls were made. A leaf of the recursion tree is an instance of the problem that is solved directly. The root of the tree represents the initial call. The depth of a node in the tree represents its level of recursion.

Lemma 3.7 *Let $U = (u_a, \ldots, u_b)$ and $V = (v_c, \ldots, v_d)$. Given an initial call of $RGD(P[u_a, u_b; v_c, v_d])$, every recursive call $RGD(P[u_q, u_r; v_s, v_t])$ has the property that for all $x \in (u_q, \ldots, u_r)$, we have that $\phi_V(x) \in (v_s, \ldots, v_t)$.*

Proof: We proceed by induction. The initial call has the property that for every $x \in U$, $\phi_V(x)$ is in V. Let us assume, by induction, that all subproblems at depth k in the recursion tree have the desired property. We show that all problems at depth $k+1$ have the desired property given that the property holds at depth k.

There are only two places in the algorithm where a recursive call takes place. Let us first look at the call in Step 4. The same argument holds for the other call in Step 2. Let $P[u_q, u_r; v_s, v_t]$ be an instance of a problem at depth k. By induction, we know that for all $x \in (u_q, \ldots, u_r)$, we have that $\phi_V(x) \in (v_s, \ldots, v_t)$. In Step 4, $P[u_q, u_r; v_s, v_t]$ is split into two instances, $P[u_q, u_k; v_m, v_p]$ and $P[u_k, u_r; v_s, v_m]$. By the crossing property, we know that for all $x \in (u_q, \ldots, u_k)$, we have $\phi_V(x) \in (v_m, \ldots, v_p)$ and for all $x \in (u_k, \ldots, u_r)$, $\phi_V(x) \in (v_s, \ldots, v_m)$. Thus, the lemma follows by induction. ∎

We are now in a position to prove the correctness of algorithm *RGD*.

Theorem 3.1 *Algorithm RGD correctly computes the restricted geodesic decomposition of chain U with respect to V.*

Proof: By Lemma 3.7, if the root of the recursion tree is an instance of *RGD*$(P[u_a, u_b; v_c, v_d])$ with the property that for all $x \in (u_a, \ldots, u_b)$, we have that $\phi_V(x) \in (v_c, \ldots, v_d)$, then all recursive calls, i.e. all other nodes of the tree, have this property. Therefore, the correctness of the restricted geodesic decomposition of U with respect to V rests on the correctness of the leaves of the recursion tree, that is, the instances of *RGD* that solve the problem directly.

If the leaf instance is degenerate, then the problem is solved directly in Step 1. The correctness of this step is proved in Lemma 3.3. If the problem is solved directly in Step 2 (in the first *if* statement), then the correctness is verified by the crossing property. Similarly, if the problem is solved directly in the first part of Step 3, the correctness is guaranteed by the crossing property. Finally, if the problem is solved directly by computing upper envelopes in Step 3, then it is correct by Lemma 3.6. Since, we have shown that all instances where the problem is solved directly are correct, the theorem follows. ■

We now turn our attention to the complexity analysis of algorithm *RGD*. We show that the algorithm runs in $O(n \log n)$ time and uses $O(n)$ space. To do this, we first show that there are $O(\log n)$ levels of recursion. Then we show that an instance of *RGD*$(P[u_a, u_b; v_c, v_d])$ (excluding recursive calls and sorting of partition points) runs in time proportional to the size of $P[u_a, u_b; v_c, v_d]$. Finally, we show that the total size of all polygons at a particular level of recursion is $O(n)$. The main ideas in the complexity analysis to follow stem from the analysis given in Suri[36].

Lemma 3.8 *Algorithm RGD$(P[u_a, u_b; v_c, v_d])$ runs in time proportional to the size of $P[u_a, u_b; v_c, v_d]$, excluding recursive calls and sorting partition points.*

Proof: Let m be the size of $P[u_a, u_b; v_c, v_d]$. Step 1 runs in time $O(m)$, by Lemma 3.3. A furthest neighbor of a point in $P[u_a, u_b; v_c, v_d]$ can be found in $O(m)$ time using the algorithm of [16]. Therefore, the first part of Step 2 runs in $O(m)$ time. Because of the structure of $P[u_a, u_b; v_c, v_d]$, constructing and triangulating the two subpolygons in the second part of Step 2 (in the *Else* statement) and in Step 4 can be done in $O(m)$ by a simple algorithm in [36] or a more complex algorithm of Chazelle [8]. Since we are excluding the sorting of partition points, Step 2 and Step 4 can be done in $O(m)$. Finally, Step 3 can be achieved in $O(m)$ time as proved in Lemma 3.6. The lemma follows. ■

We first show that the number of *distinct* edges among all the polygons constructed by algorithm *RGD* is $O(n)$, where n is the size of the polygon in the initial invocation. Recall that $P[u_a, u_b; v_c, v_d]$ denotes the region of P obtained by joining the counterclockwise chain of ∂P from u_a to u_b and the clockwise chain of ∂P from v_c to v_d with $\pi(u_a, v_c)$ and $\pi(u_b, v_d)$ (see Figure 6). We refer to $\pi(u_a, v_c)$ and $\pi(u_b, v_d)$ as *connecting paths.* There are three types of edges in $P[u_a, u_b; v_c, v_d]$. An edge that belongs to ∂P is a *primary edge*, an edge that is a subedge of an edge belonging to ∂P is a *partition edge*, and an edge that belongs to a connecting path is a *connecting edge.* The number of distinct primary edges is $O(n)$ since all primary edges are contained in the initial polygon. We now show that there are $O(n)$ distinct partition edges and connecting edges.

Lemma 3.9 *Let B be a simple polygon. Let a, c be two arbitrary but fixed points on the boundary of B and let b, d be two other points on the boundary of B such that a, b, c, d appear in this order in a counterclockwise traversal of the boundary of B. Then, all edges of $\pi(b, d)$, except perhaps three, belong to $E(a) \cup E(c)$, where $E(\alpha)$ denotes the set of edges in the shortest path tree of B from the point α.*

Proof: The proof is identical to the proof of Lemma 4 in Suri [36], except there are three edges rather than one that do not belong to $E(a) \cup E(c)$ since we consider points on the boundary of the polygon whereas Suri was dealing with vertices of the polygon. ■

Lemma 3.10 *The total number of partition points added is $O(n)$ where n is the size of the initial instance of $RGD(P[u_a, u_b; v_c, v_d])$.*

Proof: Let $P[u_a, u_b; v_c, v_d]$ represent the initial n vertex polygon. We have $U = (u_a, \ldots, u_b)$ and $V = (v_c, \ldots, v_d)$. Let n_u represent the number of vertices in the U chain and n_v the number in the V chain. We refer to an instance of $RGD(P[u_q, u_r; v_s, v_t])$ where $[u_q, u_r]$ is an edge of P and $(v_s, \ldots, v_t)$ is a polygonal chain belonging to ∂P as a *partition instance.*

Notice that a vertex in the V chain can appear in only two partition instances, since each time during the execution of *RGD* that the V chain is divided, only the dividing vertex appears in common in the two ensuing subinstances. Therefore, we can conclude that at most $2n_u$ vertices from the V chain are considered among all partition instances.

Partition points are created by extending the edges in the shortest path between a vertex v_i in the V chain and a vertex u_j in the U chain. So the number of partition points introduced is bounded by the number of distinct edges of all the shortest paths considered to create the partition points. By Lemma 3.9, all but three edges of $\pi(v_i, u_j)$ appear in $E(u_a) \cup E(u_b)$. The size of $E(u_a) \cup E(u_b)$ is $O(n)$. Since at most $2n_u$ vertices of the V chain are considered, there are at most $3 \cdot 2n_u \in O(n)$ edges not accounted for by $E(u_a) \cup E(u_b)$. Therefore, a total of $O(n)$ partition points are introduced. ■

Since a total of $O(n)$ partition points are added given that the size of the initial instance of $RGD(P[u_a, u_b; v_c, v_d])$ is n, we conclude that the total time spent sorting all partition points is $O(n \log n)$. Therefore we have the following.

Lemma 3.11 *Given an initial instance of $RGD(P[u_a, u_b; v_c, v_d])$ of size n, the total time spent sorting partition points is $O(n \log n)$.*

Lemma 3.12 *There are $O(\log n)$ levels of recursion where n is the size of the initial instance of $RGD(P[u_a, u_b; v_c, v_d])$.*

Proof: Let $\Pi = (i_1, i_2, \ldots, i_m)$ represent the the longest root to leaf path in the recursion tree. Each i_k of the path Π represents the problem instance occurring at recursion level k along the path. On any root to leaf path, there can be only one partition instance. Let us suppose that Π has a partition instance and let i_p represent it. The argument is similar if Π does not have a partition instance.

From i_1 to i_p, at each step, the U chain is divided in half as seen in Step 4 of algorithm *RGD*. Therefore, there are $O(\log n)$ instances from i_1 to the partition instance. At the partition instance i_p, by Lemma 3.10, at most $O(n)$ points are introduced. Again, from i_p to i_m, at each step the partitioned edge is divided in half as seen in Step 2. So, the length of the path from i_p to i_m is also $O(\log n)$. Therefore, Π has length $O(\log n)$. Since the longest root to leaf path in the recursion tree has length $O(\log n)$, there are $O(\log n)$ levels of recursion. ■

Lemma 3.13 *There are $O(n)$ distinct edges among all polygons constructed by $RGD(P[u_a, u_b; v_c, v_d])$.*

Proof: By Lemma 3.12, the height of the recursion tree is $O(\log n)$ where n is the size of $P[u_a, u_b; v_c, v_d]$. Since the recursion tree is a binary tree, there are $O(n)$ nodes in the tree. This means that at most $O(n)$ polygons are constructed in total. Since each polygon has two connecting paths, at most $O(n)$ connecting paths are constructed in total.

Now, a connecting path joins a point u_i on the U chain to a point v_j on the V chain. By Lemma 3.9, all but three edges of $\pi(u_i, v_j)$ appear in $E(u_a) \cup E(u_b)$. The size of $E(u_a) \cup E(u_b)$ is $O(n)$. Since at most $O(n)$ connecting paths are constructed, there are at most $O(n)$ edges not accounted for by $E(u_a) \cup E(u_b)$. This adds up to a total of $O(n)$ distinct connecting edges. By Lemma 3.10, there are only $O(n)$ distinct partition edges. By definition, there are only $O(n)$ distinct primary edges. The lemma follows. ■

All that remains to be shown is that the summed complexity of all the polygons constructed in one particular level of recursion is $O(n)$. To do this, we show that a distinct edge can belong to only a constant number of polygons in a particular level of recursion. By the construction of polygons in Step 2 and Step 4, we see that partition edges and primary edges cannot occur in two polygons at the same level of recursion. This follows from the way the U chain and V chain are divided. We now show that a connecting edge can only occur in a constant number of polygons on the same level of recursion.

In order to show this, we must consider the connecting edges as directed. All connecting paths are directed from the U chain to the V chain. Therefore, the edges of the connecting paths are arcs that are directed from one chain to the other. Consider the two paths in Figure 10. Both are connecting paths from the U chain to the V chain, and both have the edge e in common. However, e is directed one way in one of the paths and the opposite way in the other. This distinction is important in the analysis to follow.

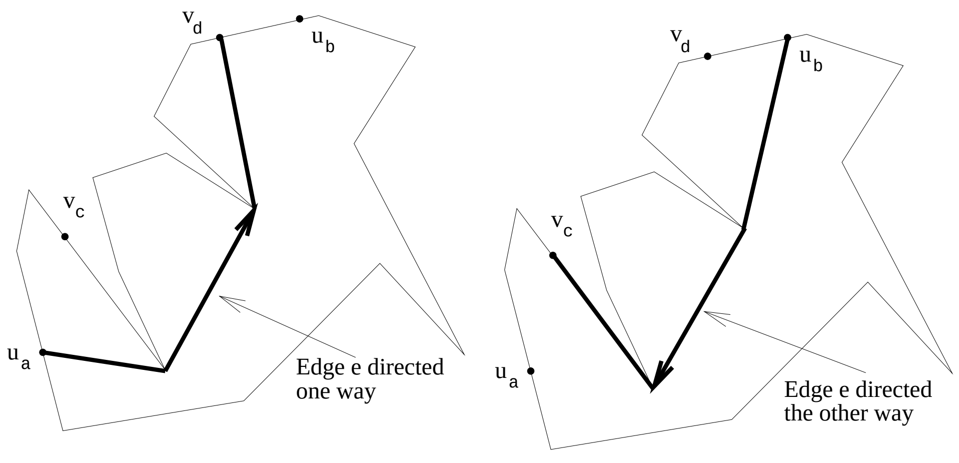


Figure 10: Directed edges must be considered.

Lemma 3.14 *[36] Let $a_1, a_2, a_3, b_1, b_2, b_3$ be six points in this order in a counterclockwise traversal of P. Suppose that the directed shortest paths $\pi(a_1, b_1)$ and $\pi(a_3, b_3)$ have a directed edge e in common. Then, the same directed edge e also is included in the shortest path $\pi(a_2, b_2)$.*

We only need to consider non-degenerate polygons at the same level of recursion. Given a degenerate polygon at recursion level i, algorithm *RGD* solves the problem directly at this stage. Therefore, since the degenerate polygon is derived from a non-degenerate polygon at level $i-1$, the complexity of the degenerate polygon can be accounted for by the non-degenerate 'parent'.

Lemma 3.15 *Let $P[u_{a_1}, u_{b_1}; v_{c_1}, v_{d_1}]$, $P[u_{a_2}, u_{b_2}; v_{c_2}, v_{d_2}]$, and $P[u_{a_3}, u_{b_3}; v_{c_3}, v_{d_3}]$ be three non-degenerate polygons that occur at the same level of recursion, such that $a_1 \leq b_1 \leq a_2 \leq b_2 \leq a_3 \leq b_3$, and $c_3 \leq d_3 \leq c_2 \leq d_2 \leq c_1 \leq d_1$. Then, the directed connecting paths of $P[u_{a_1}, u_{b_1}; v_{c_1}, v_{d_1}]$ and $P[u_{a_3}, u_{b_3}; v_{c_3}, v_{d_3}]$ are edge-disjoint.*

Proof: The proof of this lemma is similar to the proof of Lemma 7 in [36]. Suppose that the two directed connecting paths $\pi(x_1, y_1)$ and $\pi(x_3, y_3)$ share an edge e, where $x_1 \in \{u_{a_1}, u_{b_1}\}$ and $x_3 \in \{u_{a_3}, u_{b_3}\}$. Then by Lemma 3.14, $\pi(u_{a_2}, v_{c_2})$ and $\pi(u_{b_2}, v_{d_2})$ must also share edge e, contradicting the fact $P[u_{a_2}, u_{b_2}; v_{c_2}, v_{d_2}]$ is not degenerate. ∎

Theorem 3.2 *Algorithm RGD computes the restricted geodesic decomposition of chain U with respect to V using $O(n \log n)$ time and $O(n)$ space given an input of size n.*

Proof: The correctness of the algorithm is shown in Theorem 3.1. Let $P[u_a, u_b; v_c, v_d]$ be the input polygon of size n to algorithm *RGD*. By Lemma 3.8, we know that excluding recursive calls and sorting partition points, algorithm $RGD(P[u_q, u_r; v_s, v_t])$ runs in time proportional to the size of $P[u_q, u_r; v_s, v_t]$.

The size of all the polygons constructed at the same level of recursion is $O(n)$ by Lemma 3.13 and Lemma 3.15. There are $O(\log n)$ levels of recursion. Hence, the total time spent, excluding sorting partition points, is $O(n \log n)$ By Lemma 3.11, the time to sort all partitions points is $O(n \log n)$. The theorem follows. ■

In the next section, we show how to use the restricted geodesic decomposition to solve our initial problem of computing the geodesic decomposition of the boundary.

3.3 Geodesic Center Constrained to the Boundary

To compute the geodesic decomposition of the boundary of a simple polygon, we apply the algorithm for restricted decomposition three times. The following lemma of Suri[36] provides the key.

For the following lemma, we assume that $(u_1, u_2, \ldots, u_n)$ is the counterclockwise sequence of vertices of polygon P. We let $(u_a, \ldots, u_b)$ denote the counterclockwise chain of ∂P from u_a to u_b. Let u_i be an arbitrary vertex of P. Let $u_j \in \phi(u_i)$ be a geodesic furthest neighbor of u_i, and $u_k \in \phi(u_j)$ be a geodesic furthest neighbor of u_j. It is possible that $u_i = u_k$. Let us assume, without loss of generality, that u_i, u_j, u_k is the order of these vertices in a counterclockwise traversal of P starting at vertex u_i, then we have the following lemma.

Lemma 3.16 *[36] Let u_i be an arbitrary vertex of P. Let $u_j \in \phi(u_i)$ and let $u_k \in \phi(u_j)$, such that u_i, u_j, and u_k are in this order in a counterclockwise traversal of P.*

1. *for any vertex $u_l \in (u_i, \ldots, u_j)$, there exists another vertex $u_m \in (u_j, \ldots, u_i)$ satisfying $u_m \in \phi(u_l)$,*
2. *for any vertex $u_l \in (u_j, \ldots, u_k)$, there exists another vertex $u_m \in (u_k, \ldots, u_i, \ldots, u_j)$ satisfying $u_m \in \phi(u_l)$,*
3. *for any vertex $u_l \in (u_k, \ldots, u_i)$, there exists another vertex $u_m \in (u_i, \ldots, u_j, \ldots, u_k)$ satisfying $u_m \in \phi(u_l)$,*

From the above lemma, we can conclude that to compute the geodesic decomposition of P, we simply solve the following three instances of the restricted geodesic decomposition of P.

Instance 1 The U chain is $(u_i, \ldots, u_j)$ and the V chain is $(u_j, \ldots, u_k, \ldots, u_i)$.

Instance 2 The U chain is $(u_j, \ldots, u_k)$ and the V chain is $(u_k, \ldots, u_i, \ldots, u_j)$.

Instance 3 The U chain is $(u_k, \ldots, u_i)$ and the V chain is $(u_i, \ldots, u_j, \ldots, u_k)$.

Therefore, we have the following theorem.

Theorem 3.3 *The geodesic decomposition of a simple polygon can be computed in $O(n \log n)$ time and $O(n)$ space given an input of size n.*

Once the geodesic decomposition of a polygon P has been computed, the boundary-constrained geodesic center can be computed as follows. Let $(c_1, c_2, \ldots, c_t)$ represent the polygonal chains in the geodesic decomposition of the boundary of P where $\bigcup_{i=1}^{t} c_i = P$ and for every $x, y \in c_i$, $\phi(x) = \phi(y)$. For each c_i, compute the point $x \in c_i$, with the property that the geodesic distance from x to $\phi(x)$ is smallest compared to all other points in c_i. In other words, $d_G(x, \phi(x)) = \min_{\forall y \in c_i}\{d_G(y, \phi(y))\}$. The point x is referred to as the *candidate* for the chain c_i. In fact, algorithm *RGD* already computes the candidates for each chain as seen in steps 1, 2 and 3 of the algorithm. We conclude with the following theorem.

Theorem 3.4 *The boundary-constrained geodesic center of polygon P is the candidate x^*, such that $d_G(x^*, \phi(x^*)) = \min_{\forall \text{ candidates } y}\{d_G(y, \phi(y))\}$.*

Proof: Suppose that x^* is not the boundary-constrained geodesic center of polygon P. Let z be the boundary-constrained geodesic center of polygon P. Now, z is on some chain c_i of the geodesic decomposition of P. Since it is the boundary-constrained geodesic center of polygon P, it must be the candidate for chain c_i. The geodesic distance from x^* to $\phi(x^*)$ is less than or equal to the geodesic distance from z to $\phi(z)$ by definition. If $d_G(x^*, \phi(x^*)) < d_G(z, \phi(z))$, then z cannot be the boundary-constrained geodesic center. If $d_G(x^*, \phi(x^*)) = d_G(z, \phi(z))$, then x^* is also a boundary-constrained geodesic center. Both are contradictions, thereby proving the theorem. ■

3.4 Geodesic Center Constrained to a Polygonal Region

In this section, we address the problem of computing the geodesic center of a simple polygon P constrained to lie inside a simple polygon Q, where Q is contained in P. We denote this center as $GC_Q(P)$. If Q equals P then we simply have the geodesic center of the polygon P. We can further restrict the geodesic center to lie on the boundary of polygon Q, denoted $GC_{\partial Q}(P)$. In this case, if Q equals P, then we have the geodesic center constrained to the boundary of P. The reason we differentiated the problem of computing the geodesic center constrained to the boundary from this problem is that we use the geodesic furthest point Voronoi diagram to solve this problem, but to solve the former problem, we were able to avoid computing the geodesic furthest point Voronoi diagram by modifying Suri's algorithm [36]. The arguments we use to solve this problem are similar to the arguments used to solve the Euclidean center constrained to a polygon region.

Since in this and the following subsection we make extensive use of the *GFPVD*(P), let us review a few of its properties. In order to use the algorithm of [2], we assume that no vertex is geodesically equidistant from two other vertices. This can always be guaranteed by applying a slight perturbation to the vertices if the condition is violated. Like its Euclidean counter-part, *GFPVD*(P) partitions P into cells, $V(p_i)$, such that for every point $p \in V(p_i)$, the point p_i is a furthest geodesic neighbor of p. A vertex of the *GFPVD*(P) is a point that is geodesically equidistant to three vertices furthest from it. An edge between two Voronoi vertices is either a straight edge or a hyperbolic arc. Finally, the boundary of a Voronoi cell consists of a concatenation of straight edges and hyperbolic arcs. For more geometric properties of geodesic furthest point Voronoi diagrams, the reader is referred to [2, 3].

Lemma 3.17 *[3, 28] The geodesic center of a simple polygon P lies on the midpoint of the geodesic diameter of P (GDIAM(P)) or on a vertex of the GFPVD(P).*

When the geodesic center of the polygon P lies on the midpoint of the geodesic diameter, it has a special property. Let $bis(a, b)$ represent the geodesic bisector of a and b inside P, i.e. for every point x on $bis(a, b)$, $d_G(x, a) = d_G(x, b)$. Let a, b be two points of polygon P, then we have the following.

Lemma 3.18 *If the midpoint m of $\pi(a, b)$ lies on the interior of the edge separating cells $V(a)$ and $V(b)$ of GFPVD(P), then m is the geodesic center P and $\pi(a, b)$ is the geodesic diameter of P.*

Proof: We proceed by contradiction. Suppose m is not the geodesic center, and let c be the geodesic center. The bisector $bis(a, b)$ partitions polygon P into two parts. Let P_a represent the part where $\forall x \in P_a, d_G(x, a) < d_G(x, b)$ and P_b be the part where $\forall x \in P_b, d_G(x, b) < d_G(x, a)$. If c is on $bis(a, b)$, then $d_G(c, a) > d_G(m, a)$ since m is on $\pi(a, b)$ and geodesics are unique. If $c \in P_a$ then $d_G(c, b) > d_G(m, b)$ since $\pi(c, b)$ must intersect $bis(a, b)$ at some point x by the Jordan Curve Theorem and $d_G(x, b) \geq d_G(m, b)$. Similarly, if $c \in P_b$ then $d_G(c, a) > d_G(m, a)$. Therefore, by contradiction, m must be the geodesic center. Since m is the geodesic center, it follows that $\pi(a, b)$ is a geodesic diameter. ■

Before continuing, we need a few definitions. Let a, b, c be three points in a simple polygon P. The geodesic angle $\angle abc$ is the smaller of the two angles between the first link on the geodesic path from b to a and the first link on the path from b to c. Now, consider the paths $\pi(a, b), \pi(b, c)$, and $\pi(a, c)$. There exist points a', b', and c' such that the paths $\pi(a, b)$ and $\pi(a, c)$ intersect in the path $\pi(a, a')$, the paths $\pi(b, c)$ and $\pi(b, a)$ intersect in the path $\pi(b, b')$, and the paths $\pi(c, a)$ and $\pi(c, b)$ intersect in the path $\pi(c, c')$. The three paths $\pi(a', b')$, $\pi(b', c')$, and $\pi(c', a')$ form what is known as a *geodesic triangle*, denoted $\triangle a'b'c'$ (see Figure 11). The vertices a', b', c', are the only convex vertices of the geodesic triangle and are referred to as the *peaks* of the triangle. Pollack et al.[28] proved the following property concerning geodesic triangles.

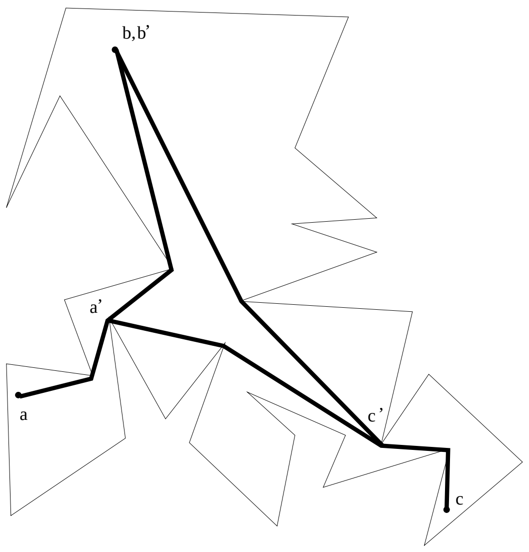


Figure 11: A Geodesic Triangle.

Lemma 3.19 *[28] If the geodesic angle $\angle ba'c$ at a' is greater than or equal to $\pi/2$, then $d_G(b,c) > d_G(a',b), d_G(a',c)$*

Lemma 3.20 *The geodesic center of P constrained to lie in Q is the midpoint m of $GDIAM(P)$ provided that m is the geodesic center of P and lies in Q.*

Proof: Follows from Lemma 3.17. ■

To address the problem of determining the location of $GC_Q(P)$ when it does not satisfy the conditions of the above lemma, we establish the following lemmas. Let a, b be two vertices of P such that they each have a corresponding cell $V(a)$ and $V(b)$, respectively, which are adjacent separated by an edge e in *GFPVD*(P). Also, $\pi(a,b)$ is not the geodesic diameter of P. Let x be a point on the interior of e, and let $\epsilon > 0$ be any small constant.

Lemma 3.21 *There exists a point $y \in e$ with $d_G(x,y) < \epsilon$ such that $d_G(y,a) < d_G(x,a)$ and $d_G(y,b) < d_G(x,b)$.*

Proof: The edge e must lie on $bis(a,b)$, since the points on e are equidistant from both a and b. The point x must be a peak of the geodesic triangle formed by the paths $\pi(x,a), \pi(x,b)$, and $\pi(a,b)$ since otherwise x would be on the path $\pi(a,b)$ which would imply that $\pi(a,b)$ was a geodesic diameter by Lemma 3.18. Also, a portion of e must be contained in the geodesic triangle, since x is on the interior of e. Let y be a point on e in the geodesic triangle. Since the geodesic angle $\angle ayb$ must be no greater than π, by Lemma 3.19 we conclude that $d_G(y,a) < d_G(x,a)$ and $d_G(y,b) < d_G(x,b)$. The lemma follows. ■

Lemma 3.22 *A point b of P cannot lie in $V(b)$.*

Proof: Let $x \in P$ be a point distinct from b. If $b \in V(b)$ then $d_G(b,b) = 0$. However, $d_G(b,x) > 0$ which contradicts the fact that $b \in V(b)$. ■

We now complete the characterization of $GC_Q(P)$.

Lemma 3.23 *If the geodesic center of P constrained to lie in Q is not the midpoint of $GDIAM(P)$, then it lies on one of the following points:*

1. *a vertex of the $GFPVD(P)$ contained in Q,*
2. *a proper intersection point of the $GFPVD(P)$ and the boundary of Q,*
3. *a vertex of the polygon Q,*
4. *a point x on an edge e of Q with the property that $\forall y \in e$, if $\phi(y) = \phi(x)$ then $d_G(y, \phi(x)) \geq d_G(x, \phi(x))$.*

Proof: If $GC_Q(P)$ does not lie on any of the points mentioned in the statement of the lemma, then it must lie in one of the regions described in the following four cases. We show that each of these cases leads to a contradiction. For simplicity of exposition, let $c = GC_Q(P)$.

Case 1: c is a point in the interior of a cell of the $GFPVD(P)$, and in $int(Q)$. Let $V(b)$ be the cell containing c. By the Jordon Curve Theorem [27], $\pi(bc)$ must intersect ∂P or $V(b)$ since $b \notin V(b)$ by Lemma 3.22. Let x be the intersection point closest to c. The point x must be in $V(b)$. Therefore b is a furthest neighbor of both x and c. However, $d_G(x, b) < d_G(c, b)$ by construction. Hence, we have a contradiction.

Case 2: c is a point in the interior of a cell of the $GFPVD(P)$, and in the interior of an edge e of Q but does not satisfy the property that $\forall y \in e$, if $\phi(y) = \phi(c)$ then $d_G(y, \phi(c)) \geq d_G(c, \phi(c))$. Since the latter property is not satisfied, a point $x \in e$ such that $\phi(x) = \phi(c)$ and $d_G(x, \phi(c)) < d_G(c, \phi(c))$ must exist. However, the very existence of x contradicts that c is the geodesic center of P constrained to lie in Q.

Case 3: c is a point in the interior of an edge e of the $GFPVD(P)$, and in $int(Q)$. Let $V(a)$ and $V(b)$ be the two cells separated by the edge e. Since c is not the midpoint of the geodesic diameter of P, by Lemma 3.21 we know that there exists a point x in e and in $int(P)$ such that $d_G(x, a) < d_G(c, a)$ and $d_G(x, b) < d_G(c, b)$. This contradicts that c is $GC_Q(P)$.

Case 4: c is a point in the interior of an edge e_v of the $FPVD(S)$, and in the interior of an edge e_p of P such that e_v and e_p intersect but not properly. Same argument as Case 3.

■

We outline the algorithm to compute $GC_Q(P)$.

Algorithm 3: *Geodesic Center of P constrained to lie in Q*

Input: A simple polygon $P = \{p_1, p_2, \ldots, p_n\}$, and a simple polygon $Q = \{q_1, q_2, \ldots, q_n\}$ with $Q \subset P$.

Output: $GC_Q(P)$

1. Compute the $GFPVD(P)$ using the algorithm of Aronov et al.[2].
2. Compute $GC(P)$ using the algorithm of Pollack et al.[28].
3. Preprocess Q in $O(n \log n)$ time for point inclusion testing in $O(\log n)$ time using the algorithms of Kirkpatrick [18] or Sarnak and Tarjan [33].
4. If $GC(P)$ is contained in Q then exit with $GC(P)$ as $GC_Q(P)$.
5. Preprocess P for shortest path queries using the algorithm of Guibas and Hershberger [15].
6. Compute the set of vertices of $GFPVD(P)$ contained in Q. Let V_c represent this set.
7. Compute the set of intersections $I_c = \{i_1, i_2, \ldots, i_k\}$ of Q with $GFPVD(P)$ using the algorithm of Chan[7].

8. Partition each edge e_i of Q such that for every pair of points $x, y \in e_i$, we have that $\phi(x) = \phi(y)$. Denote the j^{th} partition of e_i by e_{ij}.
9. For each e_{ij}, compute the point on e_{ij} closest to $\phi(e_{ij})$. If this point is not an endpoint of e_{ij}, place it in the set E_c.
10. Let P_c represent the vertices of Q.
11. Let $CAN = V_c \cup I_c \cup P_c \cup E_c$.
12. Find the set of points $G = \{x \in CAN \mid d_G(x, \phi(x)) = \min_{y \in CAN} d_G(y, \phi(y))\}$
13. Output the set G.

Notice that we assumed that the number of vertices of Q equals the number of vertices of P. Clearly, this need not be the case, however, this assumption simplifys the complexity of notation. It is quite straightforward to repeat the complexity analysis when P and Q have different cardinalities.

Theorem 3.5 *Given a polygon $P = \{p_1, p_2, \ldots, p_n\}$ and a polygon $Q = \{q_1, q_2, \ldots, q_n\}$ contained in P, we can compute the geodesic center of P constrained to lie in Q in time $O(n(n+k))$ where n is the size of the input and k is the number of intersections between the edges of the GFPVD(P) and Q.*

Proof: The correctness of the algorithm follows from Lemmas 3.20 and 3.23.

Let us analyze the complexity of the algorithm. Step 1 of the algorithm can be computed in $O(n \log n)$ time using the algorithm of Aronov et al.[2]. Step 2 can be computed in $O(n \log n)$ time using the algorithm of Pollack et al.[28]. Preprocessing for point inclusion can be done in $O(n \log n)$ using the algorithm of Kirkpatrick [18] or Sarnak and Tarjan [33]. Step 5 can be achieved in $O(n \log n)$ time by using the algorithm of Guibas and Hershberger[15]. By preprocessing the polygon for shortest path queries, in $O(\log n)$ time the geodesic distance between two points can be recovered and in $O(\log n + m)$ time the geodesic path between two points can be recovered where m is the length of the path. Step 6 can be done in $O(n \log n)$ time using the point inclusion test. Computing the intersections between *GFPVD(P)*, which consists of straight edges and hyperbolic arcs, and Q, which consists only of straight edges, can be computed in $O(n \log n + k)$ time where k is the number of intersections between Q and *GFPVD(P)* using the algorithm of Chan [7]. Once the intersection points have been computed, Step 8 can be achieved in $O(k \log n)$ time. In Step 9, to compute the point on e_{ij} closest to $\phi(e_{ij})$, we first compute the geodesic path from the endpoints of e_{ij} to $\phi(e_{ij})$ in $O(\log n + m)$ time where m is the length of the two paths using [15]. Once the two paths have been computed, finding the point geodesically closest to $\phi(e_{ij})$ can be done $O(m)$ time in the manner described in Subsection 3.2. Note that $O(m) \in O(n)$. Step 9 is executed $O(\max\{k, n\})$ times, thus the complexity is $O(n(n+k))$. Step 12 can be computed in $O(k)$ time. Therefore, the total complexity of the algorithm is $O(n(n+k))$ time. ■

3.5 Geodesic Center Constrained to a Polygonal Chain

With a slight modification, Algorithm 3 can compute the geodesic center of P constrained to lie on the boundary Q, $GC_{\partial Q}(P)$. These modifications are outlined below.

Lemma 3.24 *The geodesic center of P constrained to lie on the boundary of Q is the midpoint m of $GDIAM(S)$ provided that m is the geodesic center of P and lies on the boundary of Q.*

Proof: Follows from Lemma 3.17. ■

Lemma 3.25 *If the geodesic center of P constrained to lie on the boundary of Q is not the midpoint of $GDIAM(P)$, then it lies on one of the following points:*

1. *a vertex of the $GFPVD(P)$ on the boundary of Q,*
2. *a proper intersection point of the $GFPVD(P)$ and the boundary of Q,*
3. *a vertex of the polygon Q,*
4. *a point x on an edge e of Q with the property that $\forall y \in e$, if $\phi(y) = \phi(x)$ then $d_G(y, \phi(x)) \geq d_G(x, \phi(x))$.*

Proof: Similar case analysis as the proof of Lemma 3.23. ■

Lemma 3.24 and Lemma 3.25 completely characterize the location of $GC_{\partial Q}(P)$. The modifications to Algorithm 3 for computing these points are straightforward. Therefore, we conclude with the following.

Theorem 3.6 *Given a polygon $P = \{p_1, p_2, \ldots, p_n\}$ and a polygon $Q = \{q_1, q_2, \ldots, q_n\}$ contained in P, we can compute the geodesic center of P constrained to lie on the boundary of Q in time $O(n(n+k))$ where n is the size of the input and k is the number of intersections between the edges of the $GFPVD(P)$ and Q.*

4 Constrained Link Center

In this section, we consider the second property attributed to a good pin gate location. Recall that the second property states that the maximum number of turns that the liquid takes on its path from the pin gate to any point in the object should be small. The link metric provides a geometric interpretation of this property. The link metric measures the number of turns or bends in a path between two points. We need a few definitions about link paths before continuing.

The *link distance* between two points x and y inside a polygon P, denoted $d_L(x, y)$, is the minimum number of edges in any polygonal path connecting x and y without intersecting the boundary of P. A path $\pi_L(x, y)$ between x and y is a *minimum link path* provided that the number of edges in $\pi_L(x, y)$ is equal to $d_L(x, y)$. The *k-neighborhood* or *k-disk* about a point $x \in P$ is defined as $N_k(x) = \{y \in P \mid d_L(x, y) \leq k\}$, and the *covering radius* $c(x)$ of x is the smallest k such that $P \subset N_k(x)$. The *link radius* is defined by $r_L(P) = \min_{x \in P} c(x)$ and the *link center* of P is defined by $LC(P) = \{x \in P \mid c(x) = r_L(P)\}$. In essence, the link center is the set of points in P whose maximum link distance to any point in P is minimized, precisely the set of potential pin gates satisfying the second property of a suitable pin gate.

We review the problem of computing the link center of a simple n vertex polygon P. The problem of computing the link center was first addressed by Lenhart et al.[22] who provided a simple $O(n^2)$ time algorithm to compute $LC(P)$. Note that the link center of P is not necessarily a point as is the case with the geodesic center of P, but the link center may in fact be a geodesically convex region contained in P. Later Djidjev et al.[10] reduced the time complexity of computing $LC(P)$ to $O(n \log n)$. Therefore, to compute the link center of a simple polygon, either of these two algorithms may be used.

The problem of computing the link center constrained to the boundary of a polygon P, denoted as $LC(\partial P)$, has not been addressed. In this section, we provide a simple algorithm to compute the set $LC^*(\partial P)$ which is a subset of $LC(\partial P)$. In some cases $LC^*(\partial P)$ is in fact be equivalent to $LC(\partial P)$.

4.1 Link Center Constrained to the Boundary

In this section, we provide a simple algorithm for computing $LC^*(\partial P)$, which is a subset of $LC(\partial P)$. The following very simple observations form the basis of the algorithm.

Observation 4.1 *If $LC(P) \cap \partial P$ is non-empty, then $LC(\partial P) = LC(P) \cap \partial P$.*

Observation 4.2 *If a point $z \notin LC(P)$ is visible from a point $x \in LC(P)$, then $c(z)$ is one greater than $c(x)$.*

Two points x, y in polygon P are said to be *visible* provided that the line segment $[x, y]$ is in P. Given a set of points X in polygon P, the *strong visibility set* of X in P is $\{z \in P \mid \forall\, x \in X, [xz] \in P\}$ and the *weak visibility set* of X in P is $\{z \in P \mid \exists\, x \in X, [xz] \in P\}$. If X happens to be a simple polygon inside P, Ghosh [14] has shown that the *weak visibility set* of X in P is also a simple polygon, referred to as the *weak visibility polygon* of P from X and denoted by $WVP(X, P)$. We now outline the algorithm.

Algorithm 4: *Compute* $LC^*(\partial P)$

```
1. Compute $LC(P)$.
2. Let $LC_1 = LC(P) \cap \partial P$. If $LC_1$ is non-empty, exit with $LC_1$.
3. Compute the weak visibility polygon of $P$ from $LC(P)$.
4. Let $LC_2 = WVP(LC(P), P) \cap \partial P$. Exit with $LC_2$.
```

If LC_1 is non-empty, then $LC(\partial P)$ is equal to LC_1. If on the other hand, LC_1 is empty, then the set LC_2 must be a subset of $LC(\partial P)$ since the link center of the polygon is contained strictly in the interior of P and the covering radius of every point in LC_2 is one greater than the covering radius of a point in the link center. The complexity of the algorithm is dominated by Step 1 which can be computed in $O(n \log n)$ time using the algorithm of [10]. A simple modification to the algorithm in [10] is needed to compute the intersection of $LC(P)$ with the boundary of P in the same time complexity. Step 3 can be performed in $O(n)$ time using the algorithm of [14]. The parts of $WVP(LC(P), P)$ that are part of the boundary of P can be identified during the computation of the weak visibility polygon. Therefore, we conclude with the following theorem.

Theorem 4.1 *$LC^*(\partial P)$ can be computed in $O(n \log n)$ time.*

5 Discussion

Of the solutions presented in this paper, computing the Euclidean center, with or without constraints, as well as the link center, with or without constraints, are both conceptually and computationally simpler than computing the geodesic center. However, the Euclidean center may not always be a good candidate for the location of a pin gate as pointed out in Section 3. The link center considered alone may also not be a suitable candidate since liquid inside a mold does not necessarily travel along a link path. Combining these two constraints may provide a better approximation (e.g. computing the Euclidean center constrained to lie in the link center).

The geodesic center, although computationally more expensive, seems to be a better measure in terms of the distance the liquid travels inside a mold. A combination of the link and geodesic centers may reap the benefits of both properties of an ideal pin gate location being satisfied. For example, computing the geodesic center constrained to lie in the link center may provide a better solution than considering the geodesic center by itself.

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