CONVEX HULLS FOR RANDOM LINES

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ABSTRACT. Consider n i.i.d. random lines in the plane defined by their slope and distance from the origin. The slope is uniformly distributed on $[0, 2\pi]$ and independent of the distance R from the origin. These lines define a set $\mathbf{I}$ of $n(n-1)/2$ intersection points. It was recently shown by Atallah (1986) and Ching and Lee (1985) that the cardinality of the convex hull of these intersection points is $O(n)$, and they exhibited an $O(n \log n)$ time algorithm for computing such a convex hull. Let N_{ch} and N_{ol} be the number of points on the convex hull and outer layer of $\mathbf{I}$, respectively. We show that there exist arrangements of lines in which $N_{ol} = n(n-1)/2$. We show that, nevertheless, both N_{ch} and N_{ol} have expected values $O(1)$, and give bounds that are uniform over all distributions of R with $0 < \mathbf{E}R < \infty$. These results lead to an algorithm for computing the convex hull of $\mathbf{I}$ in $O(n \log n)$ worst-case time and $O(n)$ expected time under these conditions.

KEYWORDS AND PHRASES. Convex hull, random lines, symmetric distribution, stochastic geometry, probability inequalities, computational complexity, computational geometry.

CR CATEGORIES: 3.74, 5.25, 5.5.

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1. Introduction.

Let $\mathbf{L} = \{L_1, \ldots, L_n\}$ be a finite set of lines in the plane where each line L_i is specified by an equation $y = a_i x + b_i$ for some real numbers a_i, b_i, $1 \le i \le n$. $\mathbf{L}$ induces a partition of the plane known as the **arrangement** $A(\mathbf{L})$, into $O(n^2)$ faces, edges, and vertices. The **vertices** are the points where the lines in $\mathbf{L}$ intersect. Let V_{ij} denote the intersection point of L_i and L_j. The set $\mathbf{I} = \{V_{ij} | 1 \le i, j \le n\ , i \ne j\}$ collects all such intersection points. The **edges** are the connected components of the lines that remain when the vertices are deleted. The **faces** are the connected components of the complement of the union of the lines $L_1, L_2, \ldots, L_n$. For a detailed fundamental treatment of arrangements the reader is referred to Grunbaum (1967) and Edelsbrunner (1987). A survey of recent research results can be found in Edelsbrunner, Guibas and Sharir (1990). Figure 1 illustrates an arrangement and the convex hull of $\mathbf{I}$.

Put Figure 1 about here.

Definition: The intersection point V_{ij} for $i \ne j$ and $1 \le i, j \le n$ is said to be **extreme** with respect to line L_i if all other intersection points on L_i lie to one side of V_{ij}.

Definition: The intersection point V_{ij} for $i \ne j$ and $1 \le i, j \le n$ is said to be **critical** if V_{ij} is extreme with respect to both L_i and L_j.

It is well known that the convex hull of n points in the plane can be computed in $O(n \log n)$ worst-case and linear expected time (Devroye and Toussaint, 1981) under certain assumptions on the distribution of the points. Straightforward application of such algorithms to all the points in $\mathbf{I}$ thus leads to algorithms with $O(n^2 \log n)$ worst-case time and $O(n^2)$ space. Surprisingly, Atallah (1986) and Ching and Lee (1985) independently presented an $O(n \log n)$ worst-case time algorithm with $O(n)$ space for this problem. They show that the vertices of the convex hull of $\mathbf{I}$ are a subset of the critical points determined by pairs of lines that are adjacent on a list in which they are sorted by slope. Therefore they first sort the lines by slope to obtain $O(n)$ critical points and subsequently find their convex hull with any $O(n \log n)$ algorithm. We show here that if we chose our convex hull algorithm carefully we can obtain an algorithm which will also exhibit $O(n)$ expected complexity under a natural definition of a random line and almost any radially symmetric distribution on its parameters as well as under a model of computation that allows us to compute **floor** and **ceiling** functions in constant time.

2. The main result.

Let $X_1, \ldots, X_n$ be i.i.d. random variables with a radially symmetric distribution in the plane, i.e. X_1 is distributed as $(R\cos\Theta, R\sin\Theta)$, where Θ is uniformly distributed on $[0, 2\pi]$, and R is independent of Θ and has a given distribution on the positive reals. With each point we associate a line passing through the point and perpendicular to the segment linking the origin with the point. We thus obtain n random lines defining $\binom{n}{2}$ intersection points V_{ij} (V_{ij} is the intersection of the lines through X_i and X_j respectively).

The object of this note is to study the convex hull and the outer layer formed by the V_{ij}'s. Recall that the outer layer is the subset of V_{ij}'s with the property that an observer sitting on V_{ij} cannot see any V_{lk} in at least one of the four quadrants around him. The quadrants in the definition are aligned with the coordinate axes. Also, any convex hull point is an outer layer point.

Let N_{ch} and N_{ol} be the number of points on the convex hull and outer layer of the V_{ij}'s. We prove the following

THEOREM 1. *Let R have a distribution with $0 < \mathbf{E}R < \infty$. Then there exists a universal constant γ such that uniformly over all n, $\mathbf{E}N_{ch} \le \mathbf{E}N_{ol} \le \gamma < \infty$. The constant γ does not depend upon the distribution of R.*

The proof of Theorem 1 is given in section 4. To keep the proof relatively short, we will not try to find the best possible γ, although $\mathbf{E}N_{ch} \to 4$ as $n \to \infty$ is conjectured. The point is that $\mathbf{E}N_{ch}$ is $O(1)$, and that the bound is in fact distribution-free. The constant γ that emerges from our proof is

$$4 + \sum_{j=0}^{\infty} 2^{j+3} \exp\left(-(1-(7/6)e^{-1/6})(3/\pi)2^{j-1}\right) = 1939.4634\ldots\ .$$

The asymptotics for our problem are not unlike those for samples of n i.i.d. points in the plane having a common heavy-tailed radially symmetric distribution. The study of the size of random convex hulls dates back to Rényi and Sulanke (1963), and that of radially symmetric distributions to Carnal (1970). Additional results were obtained in the radially symmetric case by Dwyer (1988,1989) and Borgwardt (1980). Dwyer also points out the importance of the results in computational geometry, while both Borgwardt illustrates the use of convex hulls in analyzing linear programming algorithms. For a survey of results on random convex hulls, see Buchta (1985) or Schneider (1988).

The Theorem proved here can be generalized to other models, and in particular models in which the X_i's are i.i.d. but not necessarily radially symmetric. One that seems particularly promising is that in which the X_i's have a common but arbitrary density. Such generalizations will not be considered in the present paper. Additionally, it turns out that $\mathbf{E}N_\theta < \infty$ as well, where $\theta \in (0, \pi]$ is a given camera angle, and N_θ is the number of V_{ij}'s for which an observer sitting on V_{ij} with a camera of viewing angle θ can rotate his camera in such a way that at one point in the rotation, he does not see any V_{kl} through his finder. Such V_{ij}'s are called θ-visible. Every outer layer point is $\pi/2$-visible. Every π-visible point is a convex hull point. Thus, $N_\pi \le N_{ch} \le N_{ol} \le N_{\pi/2} \le N_\theta$ where $\theta \le \pi/2$. The proof that $\mathbf{E}N_\theta < \infty$ is not included here.

3. Algorithm and discussion.

Consider n i.i.d. random lines in the plane defined by their slope and distance from the origin. The slope is uniformly distributed on $[0, 2\pi]$ and independent of the distance R from the origin. Let N_{ch} and N_{ol} be the number of points on the convex hull and outer layer (maximal vectors) of $\mathbf{I}$, respectively. We show that there exist arrangements of lines in which $N_{ol} = n(n-1)/2$. By Theorem 1, nevertheless, both N_{ch} and N_{ol} have expected values $O(1)$. Therefore, if we first sort the lines using distributive partitioning (Devroye and Klincsek, 1980) in $O(n \log n)$ worst-case time, $O(n)$ expected time and $O(n)$ space (which is possible by the uniformity of the distribution of the slopes), and subsequently find the convex hull of the adjacent critical points in $\mathbf{I}$ using the output-sensitive algorithm of Kirkpatrick and Seidel (1986) which has complexity $O(k \log h)$ where k is the number of input points and h is the cardinality of the convex hull, we obtain an algorithm for computing the convex hull of $\mathbf{I}$ in $O(n \log n)$ worst-case time, $O(n)$ expected time, and $O(n)$ space.

For an arbitrary set of k points in the plane it is well known that computing the outer layer has the same complexity as computing their convex hull. However, for the case of arrangements of n lines computing the outer layer is more difficult. The standard method based on sorting $\mathbf{I}$ clearly yields an algorithm with $O(n^2 \log n)$ worst-case time, $O(n^2)$ expected time and $O(n^2)$ space. While we can improve slightly on the actual running time, it is difficult to improve on the above complexities for the following reason.

LEMMA 2. *An arrangement of the lines* $\mathbf{L} = \{L_1, L_2, \ldots, L_n\}$ *may contain as many as* $\binom{n}{2}$ *outer layer points.*

PROOF. We provide a construction illustrated in Figure 2 that has all points in $\mathbf{I}$ as outer layer points. Let the first point on $\mathbf{I}$ be a point $\mathbf{a}$ located at the origin. We accomplish this by defining line L_1 to have the equation $y = -x$, and line L_2 to have the equation $y = -(1+\epsilon)x$ where ϵ is a suitably small positive constant. Let θ denote the angle that the line with maximum slope makes (over all lines drawn so far) with the y-axis measured in a counter clockwise manner. The third line has equation $y = -(1+2\epsilon)x - \delta$, where δ is a small positive constant. This constitutes the three-step initialization phase of the construction and yields a triangle $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ such that all three vertices are outer layer points and $\mathbf{c}$ has maximum y coordinate. We now show how to add, at step r, line L_r, and create $r-1$ new outer layer points at each step. Let $\mathbf{d}$ define the leftmost intersection point on L_1 of a horizontal line colinear with $\mathbf{c}$. Line L_4 is constructed to pass through point $\mathbf{d}$ and make an angle $\theta/2$ with the y-axis. This procedure is repeated until all n lines have been used up. For example, after line L_4 has been inserted, $\mathbf{f}$ is the intersection point with maximum y-coordinate and $\mathbf{g}$ is the point on line L_1 with y-coordinate equal to that of $\mathbf{f}$. Therefore, line L_5 passes through $\mathbf{g}$ and makes an angle $\theta/2$ with the y-axis, where θ is the angle made by line L_4 with the y-axis. Since at each step the angle with the y-axis is decreased by half of the remaining angle all the lines in the arrangement have negative slope and this ensures that when we add a new line no new intersection point is dominated by any other new intersection point. By making each line pass through the point on line L_1 that has the same y-coordinate of the highest intersection point created thus far we ensure that no new intersection point is dominated by any old intersection point. Therefore at each step all the new intersection points introduced are outer layer points. Therefore all $n(n-1)/2$ intersection points are outer layer points. $\square$

Put Figure 2 about here.

It follows that even if we use the output-sensitive algorithm of Kirkpatrick and Seidel (1985) with $O(k \log v)$ complexity, where k is the number of input points and v is the number of outer layer points found, our results imply an adaptive algorithm with $O(n^2 \log v)$ actual running time, $O(n^2)$ expected time, and $O(n^2)$ space.

4. Proof of Theorem 1.

It suffices to show the result for $\mathbf{E}N_{ol}$. In our proof, we require many auxiliary results.

LEMMA 3. *Let (R_1, Θ_1) and (R_2, Θ_2) define two random lines according to the model of the previous section. Then, for $r > 0$, if the intersection point is at distance Δ from the origin,*

$$\mathbf{P}\{\Delta \geq r\} \leq \frac{2\mathbf{E}R}{r} \ .$$

PROOF. Consider Figure 3 below.

Put Figure 3 about here.

Figure 3 shows two random lines with $\Theta_1 = 3\pi/2$ and $\Theta_2 \in (0, \pi/2)$. The distribution of Δ remains unchanged if we were to replace Θ_2 by $\pi - \Theta_2$. To treat the case $\Theta_2 \in (\pi, 2\pi)$, we note that if Θ_2 is replaced by either $2\pi - \Theta_2$ or by $\pi + \Theta_2$, then the new Δ^* (say) is not greater than Δ. Thus, to bound $\mathbf{P}\{\Delta \geq r\}$ from above, it suffices to consider $\Theta_1 = 3\pi/2$ and Θ_2 uniformly distributed on $(0, \pi/2)$.

The angles η_1 and η_2 are as shown in Figure 3. We have $\eta_1 = \arcsin(R_1/\Delta)$, $\eta_2 = \arcsin(R_2/\Delta)$, and $\pi/2 - \Theta_2 = \eta_1 + \eta_2$. Thus,

$$\frac{R_1}{\Delta} = \sin \eta_1 > \frac{2}{\pi}\eta_1,$$

and similarly for R_2. Combining this shows that

$$\frac{R_1 + R_2}{\Delta} \geq \frac{2}{\pi}(\eta_1 + \eta_2) = 1 - \frac{2}{\pi}\Theta_2 = U$$

where U is a uniform $[0, 1]$ random variable. Thus, $\Delta \leq (R_1 + R_2)/U$, and by Markov's inequality, after conditioning on $R_1 + R_2$ and then un-conditioning,

$$\mathbf{P}\{\Delta \geq r\} \leq \mathbf{P}\left\{U \leq \frac{R_1 + R_2}{r}\right\} \leq \frac{2\mathbf{E}R_1}{r} \ . \ \square$$

LEMMA 4. *Let g_i be nonnegative nonincreasing functions. If $(X_1, \ldots, X_n)$ is a multinomial random variable, then*

$$\mathbf{E}\prod_i g_i(X_i) \leq \prod_i \mathbf{E}g_i(X_i) \ .$$

PROOF. The multinomial distribution is negatively quadrant dependent (see e.g., Mallows, 1968). In addition, it is S-MRR2 (it is strongly multivariate reverse of order 2), from which Lemma 4 follows according to Corollary 2.1 on page 505 of Karlin and Rinott (1980). $\square$

LEMMA 5. *Let $p \in (0,1)$ and positive integers l and n be given, and let*

$$(N_1, N_2, \ldots, N_{ln})$$

be a multinomial random vector with parameters n (the sample size) and $1/(ln)$ (equal probabilities). Then

$$\mathbf{E}\left(\prod_{1\le i\le n: N_i \ge 2} (1-p)\right) \le e^{-c_n pn},$$

where $c_n = 1 - (1+\frac{1}{l})e^{-\frac{n-1}{ln}}$, and $c_n \to c = 1 - (1+\frac{1}{l})e^{-1/l}$ as $n \to \infty$.

PROOF. We rewrite the product and use Lemma 4:

$$\begin{aligned}
\mathbf{E}\prod_{i=1}^{n}((1-p)I_{[N_i\ge 2]} + I_{[N_i<2]}) &= \mathbf{E}\prod_{i=1}^{n}(1-p+pI_{[N_i<2]}) \\
&\le \prod_{i=1}^{n}\mathbf{E}(1-p+pI_{[N_i<2]}) \\
&= \left(1-p+p\left(1-\frac{1}{ln}\right)^n + p\frac{n}{ln}\left(1-\frac{1}{ln}\right)^{n-1}\right)^n \\
&= \left(1-p+p\left(1-\frac{1}{ln}\right)^{n-1}\left(1-\frac{1}{ln}+\frac{1}{l}\right)\right)^n \\
&\le \left(1-p+pe^{\frac{n-1}{ln}}\left(1+\frac{1}{l}\right)\right)^n \\
&\le \exp\left(-pn\left(1-\left(1+\frac{1}{l}\right)e^{-\frac{n-1}{ln}}\right)\right)\ . \ \square
\end{aligned}$$

Next, we partition the plane up into 12 cones C_i of equal angle $\frac{2\pi}{12}$, and with center at the origin. The range of angles governed by C_i is $\left[\frac{2\pi(i-1)}{12}, \frac{2\pi i}{12}\right)$. Furthermore, every cone is divided up into precisely n small cones of equal angle range $\frac{2\pi}{12n}$. For one of these small cones C, the cone obtained by mirroring about the origin is denoted by C'.

Lemma 6. *Let (R_1, Θ_1) be a random line in which Θ_1 is restricted to a small cone C contained in C_{10}, and let (R_2, Θ_2) be a random line in which Θ_2 is restricted to $C' \subseteq C_4$. Let $V = V_{1,2}$ be the intersection point of the two lines, and let $r > 0$ be a given constant. Then*

$$\mathbf{P}\{\|V\| \geq r, V \in C_1\} \geq \frac{6n\mathbf{E}R}{\pi r}$$

if $n\mathbf{E}R/r \leq \pi/12$.

Proof. Figure 4 shows a particular collection of 12 cones. The circle of radius r centered at the origin is also shown. The small cones are not shown in figure 4 as each C_i contains precisely n of them.

Put Figure 4 about here.

The first fact needed is that V necessarily belongs to C_1 or C_7. An argument such as the one used in Lemma 3 shows that, using $\sin\eta \leq \eta$ for $\eta > 0$,

$$\frac{R_1 + R_2}{\|V\|} \leq |\Theta_1 - \pi - \Theta_2|.$$

Conditional on $X_1 \in C$ and $X_2 \in C'$, the right-hand-side of this inequality is distributed as $\frac{2\pi}{12n}|U - U'|$, where U and U' are i.i.d. random variables with the uniform distribution on $[0, 1]$. Note that $\mathbf{P}\{|U - U'| \leq x\} = 2x - x^2$ if $0 \leq x \leq 1$ so that $\mathbf{P}\{|U - U'| \leq x\} \geq \min(1, x)$. Also, on a large enough probability space, we may assume that U and U' are independent of R_1, R_2. Thus, we have, exploiting various symmetries,

$$\begin{aligned}
&\mathbf{P}\{\|V\| \geq r, V \in C_1\} \\
&\quad = \frac{1}{2}\mathbf{P}\{\|V\| \geq r, V \in C_1 \cup C_7\} \\
&\quad = \frac{1}{2}\mathbf{P}\{\|V\| \geq r\} \\
&\quad \geq \frac{1}{2}\mathbf{P}\left\{\frac{R_1 + R_2}{(2\pi/12n)|U - U'|} \geq r\right\} \\
&\quad \geq \frac{1}{2}\mathbf{E}\left(\min\left(1, \frac{(12n)(R_1 + R_2)}{2\pi r}\right)\right) \\
&\quad \geq \frac{1}{2}\min\left(1, \frac{12n\mathbf{E}R}{\pi r}\right)
\end{aligned}$$

$$
\begin{aligned}
&\qquad \text{(Jensen's and Chebyshev's inequalities)} \\
&= \frac{6n\mathbf{E}R}{\pi r} \ ,
\end{aligned}
$$

where we used the fact that $n\mathbf{E}R/r \le \pi/12$. $\square$

LEMMA 7. *Assume that R and r are related via the inequality*

$$
\frac{n\mathbf{E}R}{r} \le \frac{\pi}{12} \ .
$$

Let N denote the number of V_{ij}'s that end up with $\|V_{ij}\| \ge r$ and $V_{ij} \in C_1$ (in the notation of Lemma 6). Then

$$
\mathbf{P}\left\{N = 0\right\} \le \exp\left(-c_n \frac{3n^2\mathbf{E}R}{\pi r}\right)
$$

where $c_n = 1 - (1 + 1/6)e^{-\frac{n-1}{6n}}$.

PROOF. We consider the cone partition defined above Lemma 6. There are $12n$ small cones. We create $6n$ double cones by joining each cone with its mirror cone (about the origin). These are called B_i (B for baby cone). The indices of the baby cones that intersect C_4 are grouped into a set G (G for good indices). Observe that the cardinality of G is precisely n. Let N_i denote the number of points X_j that belong to the baby cone B_i. The probability of $[X_1 \in B_i]$ is precisely $1/6n$ for each i. The vector of N_i's is multinomially distributed with sample size n and equal probability parameters $1/6n$. If B_i has at least two X_j's, then we take the first two such X_j's (by index) and define the intersection point of their lines by W_i (W for way out there); otherwise, we artificially generate a W_i distributed as in the case of $N_i = 2$. This necessitates a larger probability space; in fact, we are using an embedding argument. The purpose of this is to ensure that the W_i's and the N_i's are independent. Clearly, if $p_i \stackrel{\text{def}}{=} \mathbf{P}\left\{W_i \in C_1, \|W_i\| \ge r\right\}$, then, by Lemma 6, for $i \in G$,

$$
p_i \ge \frac{1}{2} \times \frac{6n\mathbf{E}R}{\pi r} \stackrel{\text{def}}{=} p \ ,
$$

where the factor $1/2$ takes into account that there is a 50% probability of X_1 and X_2 ending up in opposite cones of B_i given that both belong to B_i. Thus,

$$
\begin{aligned}
\mathbf{P}\left\{N = 0\right\} &\le \mathbf{P}\left\{\cap_{i \in G : N_i \ge 2}[W_i \notin C_1] \cup [\|W_i\| < r]\right\} \\
&\le \mathbf{E}\Big(\prod_{i \in G : N_i \ge 2} (1 - p)\Big) \\
&\le e^{-c_n p n} \ . \quad \text{(Lemma 5)}
\end{aligned}
$$

Note also that in the last step, we required the inequality for $\mathbf{E}R$ given in the statement of the Lemma. $\square$

To prove Theorem 1, it is convenient to compute an upper bound for $\mathbf{E}N_{ol}$ with sample size $n+2$. In the notation of the Lemmata, we will take $r = (n+2)^2\mathbf{E}R$. However, we will delay the substitution of r until as far as possible to show why such a choice is practical. The plane is partitioned into 12 equal cones C_i. We reserve the notation $C_i(r)$ for the subset of C_i consisting of all points at distance at least r from the origin. The cardinality (number of points among the V_{kj}'s, $1 \le k, j \le n$, falling in a set) of $C_i(r)$ is denoted by $|C_i(r)|$. The intersection point of the lines defined by X_{n+1} and X_{n+2} is $V = V_{n+1,n+2}$. The following claim is crucial. If V is an outer layer point for the lines defined by $X_1, \ldots, X_n$ and $|C_2(2r)| > 0$, $|C_5(2r)| > 0$, $|C_8(2r)| > 0$, and $|C_{11}(2r)| > 0$, then $\|V\| \ge r$. To see this, we refer to Figure 5.

Put Figure 5 about here.

Cover the circle of radius r by a square of sides $2r$ and with center at the origin. Take any x in this square. We claim that all four quadrants centered at x properly contain one of the sets $C_i(2r)$ with $i \in \{2, 5, 8, 11\}$. In Figure 5, note that the points at coordinates $(2r, 0)$ and $(r\sqrt{3}, r)$ form an angle of $\pi/6$ at the origin. Thus, no point inside the square can be an outer layer point if all four $C_i(2r)$'s are nonempty.

The argument then continues as follows. Choose the largest integer M with the property that

$$2^M \le \frac{\pi}{12}\frac{r}{n\mathbf{E}R} = \frac{\pi}{12}\frac{(n+2)^2}{n} .$$

Such a choice is necessary in order to allow us to meet the inequality constraint imposed by Lemma 7.

We have

$$\begin{aligned}
\mathbf{E}N_{ol} &\le \binom{n+2}{2}\mathbf{P}\left\{V \text{ is an outer layer point for the lines } X_1, \ldots, X_n\right\} \\
&\le \frac{(n+2)^2}{2}\mathbf{P}\left\{\cup_{i\in\{2,5,8,11\}}[|C_i(2\|V\|)| = 0]\right\} \\
&\le 2(n+2)^2\mathbf{P}\left\{|C_1(2\|V\|)| = 0\right\} \quad \text{(by rotational symmetry)} \\
&\le 2(n+2)^2\Big(\sum_{j=0}^{M}\mathbf{P}\left\{\|V\| \in (r2^{-(j+1)}, r2^{-j}], |C_1(r2^{-(j-1)})| = 0\right\}
\end{aligned}$$

$$
\begin{aligned}
&\qquad + \mathbf{P}\left\{|C_1(r2^{-M})| = 0\right\} + \mathbf{P}\left\{\|V\| > r\right\}\Big) \\
&= 2(n+2)^2\Big(\sum_{j=0}^{M} \mathbf{P}\left\{\|V\| \in (r2^{-(j+1)}, r2^{-j}]\right\}\mathbf{P}\left\{|C_1(r2^{-(j-1)})| = 0\right\} \\
&\qquad + \mathbf{P}\left\{|C_1(r2^{-M})| = 0\right\} + \mathbf{P}\left\{\|V\| > r\right\}\Big) \\
&\le 2(n+2)^2\Bigg\{\exp\left(-c_n \frac{3n^2\mathbf{E}R2^M}{\pi r}\right) + \\
&\qquad \sum_{j=0}^{M} \frac{2^{j+2}\mathbf{E}R}{r}\exp\left(-c_n\frac{3n^2\mathbf{E}R2^{j-1}}{\pi r}\right) + \frac{2\mathbf{E}R}{r}\Bigg\}\ ,
\end{aligned}
$$

where we applied Lemmata 3 and 7. We resubstitute the values of M and r and recall that $c_n = 1-(7/6)e^{-\frac{n-1}{6n}} \to c \stackrel{\text{def}}{=} 1-(7/6)e^{-1/6}$. The upper bound can be bounded in turn by

$$
\begin{aligned}
2(n+2)^2\exp\left(-c_n\frac{3n^2 2^M}{\pi(n+2)^2}\right) + \sum_{j=0}^{\infty} 2^{j+3}\exp\left(-c_n\frac{3n^2 2^{j-1}}{\pi(n+2)^2}\right) + 4 \\
= \mathrm{I} + \mathrm{II} + 4\ ,
\end{aligned}
$$

which does not depend upon the distribution of R. Also, I tends to 0 with n since 2^M grows linearly with n. Finally, for all n large enough (so that c_n is positive), the sum in II is finite. In fact, we have

$$
\limsup_{n\to\infty} \mathbf{E}N_{ol} \le 4 + \sum_{j=0}^{\infty} 2^{j+3}\exp\left(-(3c/\pi)2^{j-1}\right)\ .
$$

This concludes the proof of the Theorem. □

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