Computational Geometric Aspects of Rhythm, Melody, and Voice-Leading

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Abstract

Many problems concerning the theory and technology of rhythm, melody, and voice-leading are fundamentally geometric in nature. It is therefore not surprising that the field of computational geometry can contribute greatly to these problems. The interaction between computational geometry and music yields new insights into the theories of rhythm, melody, and voice-leading, as well as new problems for research in several areas, ranging from mathematics and computer science to music theory, music perception, and musicology. Recent results on the geometric and computational aspects of rhythm, melody, and voice-leading are reviewed, connections to established areas of computer science, mathematics, statistics, computational biology, and crystallography are pointed out, and new open problems are proposed.

1 Introduction

Imagine a clock which has 16 hours marked on its face instead of the usual 12. Assume that the hour and minute hands have been broken off so that only the second-hand remains. Furthermore assume that this clock is running fast so that the second-hand makes a full turn in about 2 seconds. Such a clock is illustrated in Figure 1. Now start the clock ticking at "noon" (16 O'clock) and let it keep running for ever. Finally, strike a bell at positions 16, 3, 6, 10 and 12, for a total of five strikes per clock cycle. These times are marked with a bell in Figure 1. The resulting pattern rings out a seductive rhythm which, in a short span of fifty years during the last half of the 20th century, has managed to conquer our planet.

It is quite common to represent cyclic rhythms such as these, by time points on a circle. See for example the seminal paper by Milton Babbitt [12]. The rhythm in Figure 1 is known around the world (mostly) by the name of clave *Son*, and usually associated with Cuba. However, it is common in Africa, and probably travelled from Africa to Cuba with the slaves [200]. In West Africa it is traditionally played with an iron bell, and it is very common in Ghana where it is the timeline for the *Kpanlogo* rhythm [112]. Historically however, it goes back to at least the 13th century. For example, an Arabic book about rhythm written by the Persian scholar Safi-al-Din in 1252 depicts this accent rhythmic pattern using a circle divided into "pie slices," and calls it *Al-saghil-al-avval* [205]. In Cuba it is played with two sticks made of hard wood also called *claves* [139]. More relevant to this

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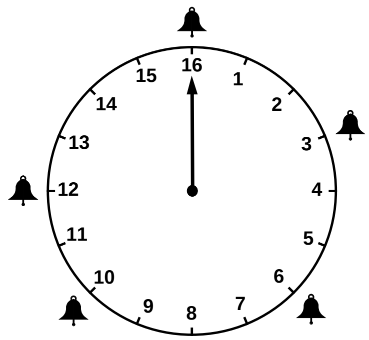
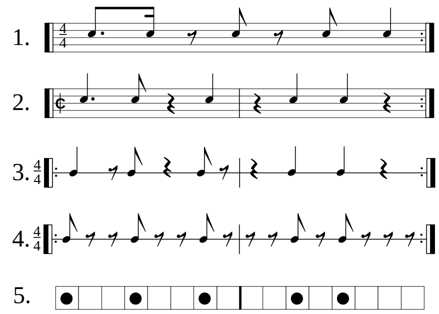


Figure 1: A clock divided into sixteen equal intervals of time.

paper, there exist purely geometric properties that may help to explain the world-wide popularity of this *clave* rhythm [183]. The word *clave*, when qualifying the rhythm rather than the instrument, assigns to it a special status as a *timeline* or *rhythmic ostinato* that functions as a key rhythmic mechanism for structuring the music that uses it.

The clave *Son* rhythm is usually notated for musicians using standard music notation which affords many ways of expressing a rhythm. Four such examples are given in the top four lines of Figure 2. The fourth line displays the rhythm using the smallest convenient durations of notes and rests. Western music notation is not ideally suited to represent African rhythm [10], [60]. The fifth and sixth lines show two popular ways of representing rhythms that avoid Western notation. The representation on line five is called the *Box Notation Method* (also *TUBS* standing for Time Unit Box System) popularized in the West by the musicologists Philip Harland at the University of California in Los Angeles and James Koetting [113]. However, such box notation has been used in Korea for hundreds of years [100]. The TUBS representation is popular among ethnomusicologists [60], and invaluable to percussionists not familiar with Western notation. It is also convenient for experiments in the psychology of rhythm perception, where a common variant of this method is simply to use one symbol for the beat and another for the pause [57], as illustrated in line six. In computer science the clave *Son* might be represented as the 16-bit binary sequence shown on line seven. Line eight depicts the adjacent interval duration representation of the clave *Son*, where the numbers denote the durations (in shortest convenient units) of the intervals between consecutive *onsets* (beginning points in time of notes). The compactness and ease of use in text, of this numerical interval-duration representation, are two of its obvious advantages, but its iconic value is minimal. Furthermore, this notation does not allow for representation of rhythms that start on a silent pulse (anacrusis). Finally, line nine illustrates the *onset-coordinate vector* notation. Here the x-axis represents time in a continuous manner starting at time zero, and the numbers indicate the x-coordinates at which the onsets occur. This representation is useful for computing dissimilarities between rhythms from the point of view of *linear assignment problems* [43], [40], [41]. Note however, that an additional piece of information is needed for some of its applications, namely, at what coordinate value the rhythm ends. For a description of additional geometric methods used to represent rhythms in both modern times and antiquity see [186], [173]. In this paper we will use notations 5 through 9, as well as other geometric representations, interchangeably depending on contextual appropriateness as well as for the sake of variety. Note that the physical lengths of the representations in the manuscript have no bearing on the duration of the corresponding rhythms in real time. and each sounded or silent pulse may be taken as one arbitrary unit of time. The important information is the length of the



6. [x . . x . . x . . . x . x . . .]

7. 1 0 0 1 0 0 1 0 0 0 1 0 1 0 0 0

8. (3 3 4 2 4)

9. (0, 3, 6, 10, 12)

Figure 2: Nine common ways of representing the *clave Son* rhythm.

cycle (timespan) and the total number of pulses in the cycle.

Rhythms are modelled in this paper as points in one-dimensional time (either on a straight line or cyclically on a circle). Melody, on the other hand, is often modelled in two dimensions: time and pitch. In such a two-dimensional space melody may be considered as a rhythm (time) in which each onset has its own y-value (pitch). However, melodies may nevertheless be analysed quite effectively in some applications areas such as music information retrieval, by ignoring the pitch information, and using only the time dimension. Such is the case for example in *query-by-tapping* systems [58], [142]. Therefore although we will often use the language of the rhythm domain, most of the results described here apply to the analysis of scales, chords, melodies, and voice-leading as well [28]. Melody is composed of notes from a scale, and scales may also be represented on a one-dimensional pitch circle as is done here with rhythms. The ubiquitous diatonic scale (determined by an octave on a standard piano) illustrated at the bottom of Figure 3, can be mapped to a circle as shown in the top of the figure, which shows the C-major triad chord as a triangle. Such chord polygons are also called *Krenek* diagrams [128], [153].

In this paper several geometric properties of musical rhythms, scales, melodies and voice-leading are analysed from the musicological and mathematical points of view. Several connecting bridges between music theory, musicology, discrete mathematics, statistics, computational biology, computer science, and crystallography are illuminated. Furthermore, new open problems at the interface of these fields are proposed. No attempt is made to provide an exhaustive survey of these vast areas. For example, we ignore the mathematics of sound [172], tuning methods [78], and the construction of musical instruments [167], [89]. We also ignore geometric symmetry transformations of musical motifs in two-dimensional pitch-time space [97]. Thus we limit ourselves to results of particular interest to the computational geometry, music information retrieval, and music theory communities. Furthermore, the illustrative rhythmic examples are restricted to a few of the most internationally well known rhythm timelines, with the hope that they will inspire the reader to comb the relevant literature contained in the references, for further details in the rhythmic as well as other music domains.

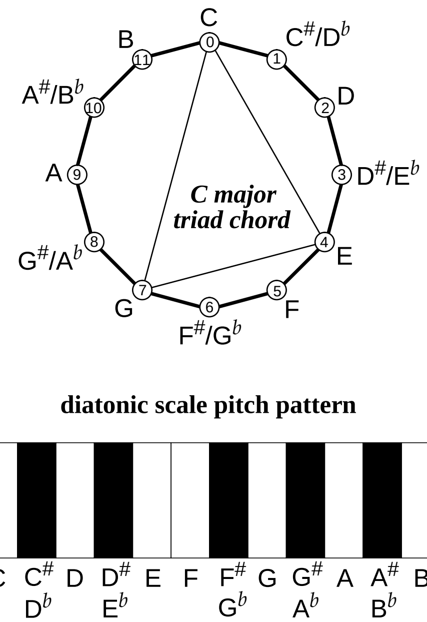


Figure 3: A chord represented as a triangle in the pitch circle.

2 Measures of Rhythmic Evenness

Consider the following 12-pulse rhythms expressed in box-like notation: [x . x . x . x . x . x .], [x . x . x x . x . x . x] and [x . . . x x . . x x x .]. It is intuitively clear that the first rhythm is more *even* (well spaced) than the second, and the second is more even than the third. In passing we note that the second rhythm is internationally the most well known of all the African timelines. It is traditionally played on an iron bell, and is known on the world scene mainly by its Cuban name *Bembé* [185]. It is also referred to in the literature as the "standard" pattern [106], [121], [7], [175]. Also noteworthy is the fact that this rhythm is isomorphic to the diatonic scale [149]. The onsets correspond to the white keys on the piano octave pictured at the bottom of Figure 3. Traditional, as well as modern, rhythm timelines have a tendency to exhibit such properties of evenness to one degree or another. Therefore mathematical measures of evenness, together with other geometric properties, serve as features with which rhythms may be compared, classified, and retrieved efficiently from music data bases. They also find applications in computational music theory [17], [189], as well as the new field of mathematical ethnomusicology [30], [188], [31], where they may help to identify, if not explain, cultural preferences of rhythms in traditional music. For example, it is highly plausible that rhythms for dancing should be very even in order to provide "drive" or "forward motion," but they should not be perfectly even, since (without other distractions) they would quickly become monotonous. Therefore maximally even rhythms such as [x . . . x . . . x . . .], which provide merely an equally spaced series of pulses ad infinitum, are not interesting from the rhythmic theoretic point of view. To make maximally even rhythms a more interesting object of investigation we need to add some constraints to our class of rhythms. One useful constraint, for example, is to make the number of onsets (k) and the number of pulses (n) in the cycle, relatively prime. The class of *Euclidean* rhythms, generated with the Euclidean algorithm for computing the greatest common divisor between two numbers k and n, are as even as possible (maximally even) without being perfectly even [189], [51].

2.1 Maximally even rhythms

In music theory much attention has been devoted to the study of intervals used in pitch scales and chords [74], [118], [165], but relatively little work has been devoted to the analysis of time duration intervals of rhythm. Indeed, almost all the attention given by scholars in the past 2500 years has been lavished on tuning systems, scales, chords, and harmony, leaving rhythm by the wayside. This situation has been rapidly changing during the past twenty years, as evidenced by the books published recently: Grosvenor Cooper and Leonard Meyer [48], Simha Arom [10], Christopher Hasty [93], Martin Clayton [36], Kobi Hagoel [88], Justin London [123], and William Sethares [162]. The book by David Temperley devotes several chapters to the topic of meter in both Western and African rhythm [175]. There has also been some work on "transferring" the analysis of pitch to rhythm [28], [149], [150], [151]. Clough and Douthett [38] introduced the notion of *maximally even sets* with respect to chords represented on a circle. According to Block and Douthett [17], Douthett and Entringer went further by constructing several mathematical measures of the amount of *evenness* contained in a chord (see the discussion on p. 41 of [17]). One of their measures simply adds all the interval arc-lengths (geodesics along the circle) determined by all pairs of pitches in the chord. This definition may be readily transferred to durations in time, of cyclic rhythms represented on a circle, as illustrated in Figure 1. However, this measure is too coarse to be useful for characterizing or comparing rhythm timelines such as those studied in [183] and [185]. Admittedly, the measure does differentiate between rhythms that differ widely from each other. For example, the two four-onset rhythms [x . . . x . . . x . . . x . . .] and [x . x . x . . x . . . .] yield evenness values of 32 and 23, respectively with the arc-length measure, reflecting clearly that the first rhythm is more evenly spaced than the second. However, *all* six of the 16-pulse clave patterns illustrated in Figure 4, and discussed in [183], have an equal evenness value of 48, and yet the *Rumba* clave is clearly more uneven than the *Bossa-Nova* clave (also called *Bossa* for short). The counter-intuitive behaviour of the sum-of-arc-lengths measure in this example is explained by the characterization of those configurations of points that yield a maximum value of the sum, recently discovered by Minghui Jiang [104]. Let two antipodal points p and q on the circle be such that neither coincides with an onset of the rhythm. These two points partition the circle into two semi-circles. Jiang showed that the maximum value of the sum-of-arc-lengths is obtained if, and only if, the configuration of points on the circle has the property that for every such pair of points p and q, the number of onsets in one semi-circle differs by at most one from the number in the other semi-circle. It turns out that all six 16-pulse clave patterns in Figure 4 satisfy this *balanced* condition, and hence all realize the maximum value of the measure.

The use of interval chord-lengths (as opposed to geodesic distances), proposed by Block and Douthett [17], yields a more discriminating measure. It is important to emphasize that in the remainder of this paper inter-onset distances will be measured in these two different ways. In both approaches a rhythm is represented as a set of points (the onsets) on a circle. In the first approach the distances are *geodesic*, the circle is viewed as a circular lattice C_n with the distances equal to the actual durations of time. Furthermore, the geodesic distance is the smaller of the two possible distances between two points on a circle. In the second approach the onsets are viewed as vertices of a convex polygon inscribed in a circle, and the distances are equal to the lengths of the edges and internal diagonals of the polygon, i.e., Euclidean distances.

2.2 Maximizing the sum of distances

The evenness measure of Block and Douthett [17], which sums all the pairwise (straight line) chord lengths of a set of points on the circle, brings up the question of which configurations of points (rhythms) achieve maximum evenness. In fact, this problem was investigated by the mathematician

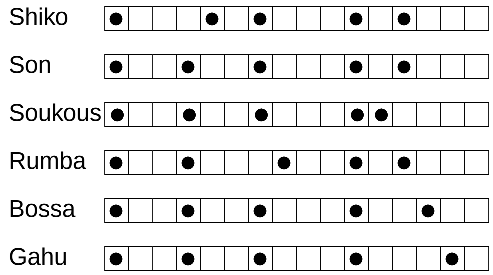


Figure 4: The six 12-pulse clave/bell patterns in box notation.

Fejes Tóth [177] almost forty years earlier, without the restriction of placing the points on the circular lattice. He showed that the sum of the pairwise distances determined by n points contained in a circle is maximized when the points are the vertices of a regular n-gon inscribed in the circle.

The discrete version of this problem, of interest in music theory [17], is also a special case of several problems studied in computer science and operations research. In graph theory it is a special case of the maximum-weight clique problem [69]. In operations research it is studied under the umbrella of obnoxious facility location theory. In particular, it is one of the *dispersion* problems called the discrete p-maxian location problem [67], [68]. Because these problems are computationally difficult, researchers have proposed approximation algorithms [92], and heuristics [68], [201], for the general problem, and have sought efficient solutions for simpler special cases [154], [170].

Fejes Tóth [177] also showed that in three dimensions four points on the sphere maximize the sum of their pairwise distances when they are the vertices of a regular tetrahedron. The problem remains open for more than four points on the sphere. For more details and references concerning the 3-dimensional problem the reader is referred to [191]. However, these excursions do not appear to be directly related to music.

In 1959, Fejes Tóth [178] asked a more difficult question by relaxing the circle constraint in the planar problem. He asked for the maximum sum of distances of n points in the plane under the constraint that the diameter of the set is at most one. Pillichshammer [144] found upper bounds on this sum but gave exact solutions only for $n = 3, 4$, and 5. For $n = 3$ the points form the vertices of an equilateral triangle of unit side lengths. For $n = 5$ the points form the vertices of a regular pentagon with unit length diagonals. For $n = 4$ the solution may be obtained by placing three points on the vertices of a Reuleaux unit-diameter triangle, and the fourth point at a midpoint of one of the Reuleaux triangle arcs. However, the four points do not lie on a circle, and hence this construction does seem directly related to music. A Reuleaux triangle is the figure obtained by intersecting three circular disks centered on three points, respectively, that are the vertices of a regular triangle, such that the radius of each disk equals the distance between two of these points. The problem remains open for more than five points in the plane. In the mathematics literature such problems have also been investigated with the Euclidean distance replaced by the squared Euclidean distance [143], [145], [204]. Again, however, these versions of the problem do not seem to be directly related to music.

2.3 The linear-regression-evenness measure

As mentioned in the preceding, Douthett and Entringer explored several mathematical measures of the amount of *evenness* contained in a chord, and one of their measures simply adds all the interval arc-lengths determined by all pairs of points on the circle. The reader may verify that

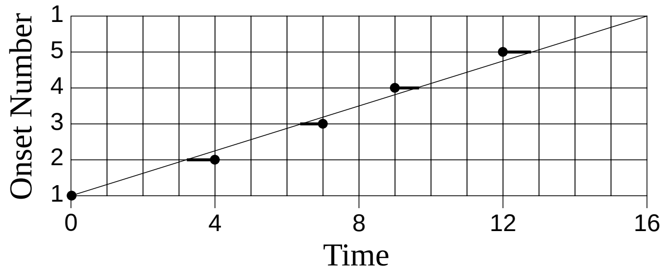


Figure 5: The linear-regression evenness measure of a rhythm.

according to this measure the *Bembé* rhythm [x . x . x x . x . x . x] is a maximally even set among all seven-onset 12-pulse rhythms [185]. For a rhythm represented as a binary sequence of length n with k onsets, the measure of Douthett and Entringer can be trivially computed in $O(n + k^2)$ time using brute force, $O(n)$ for reading the sequence and finding the coordinates of the onsets, and $O(k^2)$ for summing the pairwise arc-lengths. However, Minghui Jiang has shown that the sum of the pairwise arc-lengths may be computed in optimal $O(k)$ time [104]. Therefore the Douthett-Entringer measure may be computed in optimal $O(n)$ time. Using Euclidean lengths instead of arc-lengths, as proposed by Block and Douthett [17], of course does not change the computational complexity of the brute force method. However, whether an $O(n)$ time algorithm exists for this version of the problem along the lines of Jiang's algorithm remains an open problem. It is possible to define a different measure of rhythmic evenness which is not only very simple and also computable in $O(n)$ time, but which is sensitive enough to discriminate between all six 16-pulse clave rhythms shown in Figure 4. Such a measure is described in the following.

Michael Keith [109] proposed a measure of the *idealness* of a scale which measures the evenness of the pitch intervals present in the scale. Toussaint [184] applied Keith's idea to measure the evenness of rhythms. Consider the following 5-note rhythm on a 16-unit timespan: [x . . . x . . x . x . . x . . .]. This sequence is mapped onto a two-dimensional grid of size 16 by 5 as pictured in Figure 5. The x-axis represents the 16 units of time (pulses) at which the five onsets are played and the y-axis indexes the five onsets. The rhythm is shown in solid black circles on the 0, 4, 7, 9, and 12 time positions. The intersections of the horizontal onset-lines with the diagonal line indicate the times at which the five onsets should be played to obtain a perfectly even pattern. The deviations between these intersections and the actual positions of the onsets are shown in bold line segments. The sum of these deviations serves as a measure of the un-evenness of the rhythm. Because of its similarity to linear regression fitting of data points in statistics this measure is termed the *linear-regression-evenness* of the rhythm. Viewed as a purely mathematical curve-fitting problem, the distances from the onset points to the line may be measured in either the horizontal, vertical, or orthogonal directions. However, the horizontal direction seems more natural since we are measuring deviations in time. Note that in order to make meaningful comparisons among rhythms that contain a different number of onsets, or a different time scale, this measure of evenness would have to be normalized by dividing the score by the number of onsets, and by scaling the time span, respectively. In addition, unlike the measure that sums the Euclidean chord lengths, this measure is not rotationally invariant. This is either a drawback or a useful feature, depending on its application. If we are interested in discriminating between patterns under all possible rotations, it is clearly a flaw. However, if the patterns to be compared are rhythms fixed in time, then it is an important feature, lest the downbeats be confused with upbeats, for example.

If we want to make the linear-regression evenness measure, invariant under rotations of cyclic rhythms, then the horizontal direction is more natural for measuring the deviations because it corresponds to arc length on the time circle. Thus, the linear-regression evenness measure is equivalent to the sum of the arc-lengths on a circle, between the rhythm's k onset points and the k vertices of a regular polygon inscribed in the circle with one vertex anchored at zero. It may be readily verified that the six clave rhythms discussed in the preceding have the following values of linear-regression-evenness: *Bossa Nova* = 1.2, *Son* = 1.8, *Rumba* = 2.0, *Gahu* = 2.2, *Shiko* = 2.4 and *Soukous* = 2.8. The linear-regression-evenness measure may be computed trivially in $O(n)$ time, since k is usually very close to $n/2$, i.e., $O(n)$ [149], [150], [151]. The reader may wonder what the fuss over computational complexity is when $k = 5$ and $n = 16$, as in these clave patterns. However, when analyzing the evenness of the distributions of markers in DNA sequences, both k and n are in the thousands [111]. How useful this feature will be in applications is an open problem.

When we are interested in a cyclic rhythm regardless of its starting point then it is common to call it a *rhythmic-necklace* [187], [189], [191], [192], [8], [146], [22]. In music theory a necklace is called a *transpositional set class* [169], whereas an instance of a necklace (or just a rhythm) is called a set.

There exists a variety of methods, other than the two discussed in the preceding, for measuring evenness. For a comparison of these and other methods see [51] and [6].

3 Duration Interval Spectra of Rhythms

Rather than focusing on the *sum* of all the inter-onset duration intervals of a rhythm, or on the sum of all the inter-onset chord lengths when rhythms are represented as points on the circular lattice, as was done in the preceding section, here we examine the shape of the *spectrum* of the frequencies with which all the inter-onset durations occur. Again we assume rhythms are represented as points on a circle as in Figure 1. In music theory this spectrum is called the *interval vector* (or full-interval vector) [128]. For example, the interval vector for the clave *Son* pattern of Figure 1 is given by [0,1,2,2,0,3,2,0]. It is an 8-dimensional vector because there are eight different possible duration intervals (geodesics on the circle) between pairs of onsets defined on a 16-unit circular lattice. For the clave *Son* there are 5 onsets (10 pairs of onsets), and therefore the sum of all the vector elements is equal to ten. A more compelling and useful visualization of an interval vector is as a histogram. Figure 6 shows the histograms of the full-interval sets of all six 16-pulse clave/bell patterns pictured in Figure 4.

Examination of the six histograms leads to questions of interest in a variety of fields of enquiry: musicology, geometry, combinatorics, crystallography, and number theory. For example, David Locke [122] has given musicological explanations for the characterization of the *Gahu* bell pattern (shown at the bottom of Figure 4) as "rhythmically potent," exhibiting a "tricky" quality, creating a "spiralling effect," causing "ambiguity of phrasing" leading to "aural illusions." Comparing the full-interval histogram of the *Gahu* pattern with the five other histograms in Figure 6 leads to the observation that the *Gahu* is the only pattern that has a histogram with a maximum height of 2, and consisting of a single connected component of occupied histogram cells. The only other rhythm with a single connected component is the *Rumba*, but it has 3 intervals of length 7. The only other rhythm with maximum height 2 is the *Soukous*, but it has two connected components because there is no interval of length 2. Only *Soukous* and *Gahu* use seven out of the eight possible interval durations.

The preceding observations suggest that perhaps other rhythms with uniform (flat) histograms, and few, if any, gaps may be interesting from the musicological point of view as well. Does the histogram shape of the *Gahu* rhythm play a significant role in the rhythm's special musicological properties? If so, this geometric property could provide a heuristic for the discovery and automatic

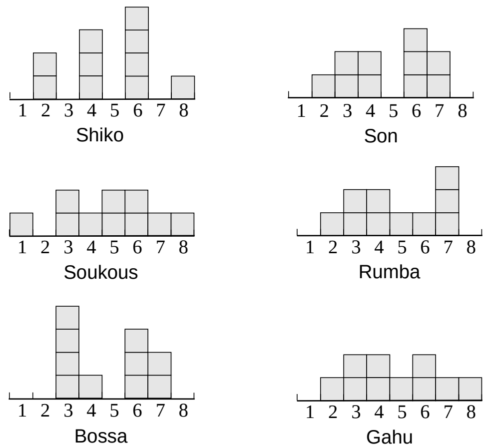


Figure 6: The full-interval histograms of the 16-pulse clave/bell patterns.

generation of other "good" rhythms. Such a tool could be used for music composition by computer. With this in mind one may wonder if rhythms exist with the most extreme values possible for these properties. Let us denote the family of all rhythms consisting of k onsets in a time span cycle of n units by $R[k,n]$. In other words $R[k,n]$ consists of all n-bit cyclic binary sequences with k one's. Thus all the 16-pulse clave/bell patterns in Figure 4 belong to $R[5,16]$.

The first natural question that arises is whether there exist any rhythms whose inter-onset intervals have perfectly flat histograms of height one with no gaps. This is clearly not possible with $R[5,16]$. Since there are only 8 possible different interval lengths and 10 distance pairs, there must exist at least one histogram cell with height greater than one. The second natural question is whether there exists an $R[5,16]$ rhythm that uses all eight intervals. The answer is yes; one such pattern is [x x . . . x . x . . . . . . x . .] with interval vector given by [1,1,1,2,1,2,1,1]. However, the rhythm [x x . . x . x . . . . .] belonging to the family $R[4,12]$ depicted in Figure 7 (a) does have a perfectly flat histogram: every one of the inter-onset intervals occurs exactly once; its interval vector is [1,1,1,1,1,1]. Such sets are also called *Golomb* rulers when the points are considered on a line rather than a circle [148], and have applications to the placement of antenas in radio-astronomy.

For a rhythm to have "drive" it should not contain silent intervals that are too long, such as the silent interval of length six in Figure 7 (a). A word is in order concerning our polygonal representation of rhythms here. Although the k-onset rhythms and the n-pulse time-spans are depicted as k-vertex and n-vertex polygons, respectively, the vertices of these polygons lie on a circle, and the numbers associated with each edge and diagonal of the polygons denote geodesic distances on the underlying circle, that represent the durations in time.

One may wonder if there are other rhythms in $R[4,12]$ with interval vectors equal to [1,1,1,1,1,1], and if they exist, are there any with shorter silent gaps. It turns out that the answer to this question is also yes. The rhythm [x x . x . . . x . . . .] pictured in Figure 7 (b) satisfies all these properties; its longest silent gap is five units. In music theory these concepts have been studied in the context of pitch, where the chords are represented on a circle, as in Figure 3. The four-note chords in Figure 7 are known as the *all-interval tetrachords*. In general, the chords that have the same interval vectors, such as the polygons in Figure 7, are often called *Z-related chords* [165], [45], [46], [47].

A cyclic sequence such as [x x . . x . x . . . . .] is an instance of a *necklace* with "beads" of two colors [109]; it is also an instance of a *bracelet*. Two necklaces are considered the same if one can be rotated so that the colors of its beads correspond, one-to-one, with the colors of the other.

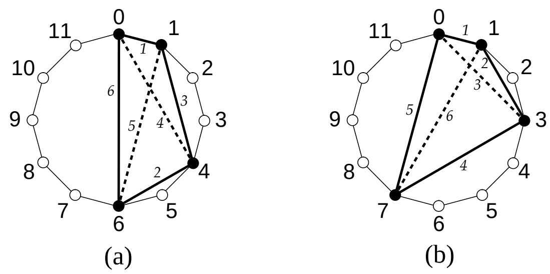


Figure 7: Two all-interval flat-histogram rhythms of height one.

Two bracelets are considered the same if one can be rotated or turned over (mirror image) so that the colors of their beads are brought into one-to-one correspondence. The rhythms in Figure 7 clearly maintain the same interval vector (histogram) if they are rotated, although this rotation may yield rhythms that sound quite different. Therefore it is useful to distinguish between rhythm-necklaces, and just plain rhythms (necklace *instances* in a fixed rotational position with respect to the underlying beat). The number of onsets in a rhythm is called the *density* in combinatorics, and efficient algorithms exist for generating necklaces with fixed density [160]. In music theory a bracelet is called a *TnI set class* [169].

3.1 Rhythms with specified duration multiplicities

In 1986 Paul Erdős [64],[65] asked whether one could find n points in the plane (no three on a line and no four on a circle) so that for every i, $i = 1, ..., n-1$ there is a distinct distance determined by these points that occurs exactly i times. Solutions have been found for $2 \leq n \leq 8$. Palásti [140] considered a variant of this problem with further restrictions: no three form a regular triangle, and no one is equidistant from three others. In 1990 Paul Erdős and János Pach [66] proposed variants of this problem with restrictions on the diameter of the set. For additional variants and open problems the reader is referred to the recent book by Brass, Moser, and Pach [20].

A musical scale whose pitch intervals are determined by points drawn on a circle, and that has a restricted version of the property specified by Erdős is known in music theory as a *deep* scale [105]. In a *deep* scale there are no zero entries in the histogram of intervals. We will transfer this terminology from the pitch domain to the time domain and refer to rhythms with this property as *deep* rhythms.

Deep scales have been studied as early as 1966 by Terry Winograd [203], and 1967 by Carlton Gamer [76], [77]. Their definition of *deep* is too restrictive for rhythms. A more useful generalization allows entries with multiplicity zero. To differentiate between the two definitions we call a musical scale or rhythm *Winograd-deep* if every possible distance from 1 to $\lfloor n/2 \rfloor$ has a unique multiplicity, where n is the total number of elements or pulses in the cycle. On the other hand, we define an *Erdős-deep* rhythm (or scale) to be a rhythm with the property that, among the histogram entries with non-zero multiplicity, for every $i = 1, 2, ..., k-1$, there is a nonzero distance determined by the onset-points on the circle that occurs exactly i times. Demaine et al., [50] characterized *Erdős-deep* rhythms, and showed that every *Erdős-deep* rhythm has a *shelling*. An *Erdős-deep* rhythm has a shelling if there exists a sequence of all its onsets such that the onsets may be deleted one at a time, so that after each deletion the resulting rhythm remains *Erdős-deep*. The most famous example of a *Winograd-deep* scale is the ubiquitous Western *diatonic* scale. Also, the *Bembé* rhythm mentioned

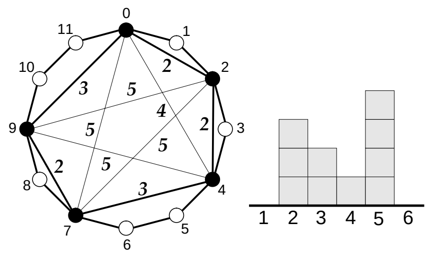


Figure 8: The *Fume-Fume* rhythm [112] (also pentatonic scale) and its inter-onset interval histogram.

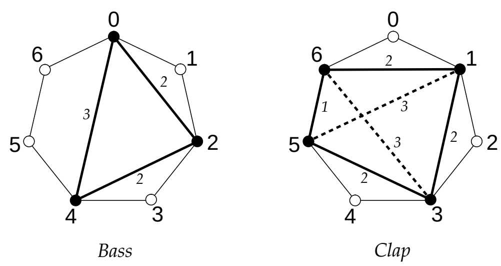


Figure 9: The bass and clap patterns of Dave Brubeck's *Unsquare Dance* are a complementary pair of *deep* rhythms.

in the preceding is of course also a *Winograd-deep* rhythm since it is isomorphic to the diatonic scale. The most famous 5-onset African rhythm timeline, the *Fume-Fume* [112], is an *Erdős-deep* rhythm. It is pictured in Figure 8 along with its inter-onset interval histogram. The reader may easily verify that deleting the third onset (at position 4) results in another *Erdős-deep* rhythm.

As a final example consider Figure 9 which illustrates the timeline pattern of Dave Brubeck's *Unsquare Dance*. It consists of two parts: the bass on the left, and the hand-clapping pattern on the right. Both parts are *deep* rhythms: the bass part is *Erdős-deep* whereas the clapping pattern is *Winograd-deep*. Furthermore, they are complementary, i.e., their union tiles the circular lattice C_7, and their intersection is empty. The bass pattern given by [x . x . x . .] is the meter of this piece and, although hardly ever used in pop music, it is common in eastern Europe and the Middle East. It is a rhythm found in Greece, Turkestan, Bulgaria, and Northern Sudan [11]. It is the *Dáwer turan* rhythmic pattern of Turkey [88]. It is the *Ruchenitza* rhythm used in a Bulgarian folk-dance [149], as well as the rhythm of the Macedonian dance *Eleno Mome* [163]. It is also the rhythmic pattern of Pink Floyd's *Money* [109]. When started on the second onset as in [x . x . . x .] it is a Serbian rhythm [11]. When started on the third onset as in [x . . x . x .] it is a rhythmic pattern found in Greece and Turkey [11]. In Yemen it goes under the name of *Daasa al zreir* [88]. It is also the rhythm of the Macedonian dance *Tropnalo Oro* [163], the rhythm for the Bulgarian *Makedonsko Horo* dance [199], as well as the meter and clapping pattern of the *tīvrā tāl* of North Indian music [36].

The question posed by Erdős is closely related to the general problem of reconstructing sets from interpoint distances: given a distance multiset, construct all point sets that realize the distance

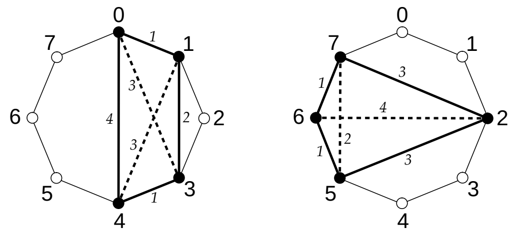


Figure 10: Two complementary homometric rhythms.

multiset. This problem has a long history in crystallography [115], and more recently in DNA sequencing [164]. Two non-congruent sets of points, such as the two different necklaces of Figures 7, are called *homometric* if the multisets of their pairwise distances are the same [158]. For an extensive survey and bibliography of this problem see [115]. The special cases relevant to the theory of rhythm, when points lie on a line or circle, have received some attention, and are called the *turnpike* problem and the *beltway* problem, respectively [115]. The term *homometric* was introduced by the crystallographer Lindo Patterson [127].

Some existing results on homometric sets on the circular lattice are most relevant to the theory of rhythm (and music theory in general). For example many drumming patterns have two sounds (such as the high and low congas) that are complementary. Similar patterns occur with double bell rhythms such as the *a-go-go* bells used in Brazilian music and the *gankogui* bell used in West African music, as well as the *paradiddle* rhythms used in snare drum technique [159]. It is known that every n-point subset of the regular $2n$-gon is homometric to its complement [115]. Musicians call this the *Babbit Hexachord Theorem*, or just *Hexachord Theorem* for short. More generally, the hexachord theorem states that two non-congruent complementary sets with $k = n/2$ (and n even) are homometric [165]. The earliest proof of this theorem in the music literature appears to be due to Milton Babbitt and David Lewin [13], [116], [117], [118]. It used heavy machinery from topology. Later Lewin obtained new proofs using group theory. Emmanuel Amiot [6] discusses some of the history regarding Lewin's proof, and shows a proof using the Discrete Fourier Transform. In 1974 Eric Regener [155] found an elementary simple proof of a more general version of this theorem. Music theorists have been unaware that this theorem was known to crystallographers about thirty years earlier [141]. It seems to have been proved by Lindo Patterson [141] around 1940 but it appears that he did not publish a proof. In the crystallography literature the theorem is called Patterson's *second* theorem [24]. The first published proof in the crystallography literature is due to Buerger [23]; it is based on image algebra, and is non-intuitive. A much simpler, more general, and elegant elementary proof by induction was later found by Iglesias [101]. Another simple elementary proof was published by Steven Blau in 1999 [16]. Marjorie Senechal recently found what may be the simplest proof of this theorem [161]. An expository survey of elementary proofs of the generalized hexachord theorem is under preparation [193].

The hexachord theorem leads immediately to a simple method for the generation of two-tone complementary rhythms in which each of the two parts is homometric to the other. One example is illustrated in Figure 10. It is also known that two rhythms are homometric if, and only if, their complements are [32]. This concept provides another, as yet unexplored, tool for music composition by computer.

3.2 Rhythms with specified numbers of distinct durations

The histograms of the rhythms illustrated in Figure 6 reveal another important parameter of rhythms: the number of *distinct* inter-onset durations contained in a rhythm. Clearly, the larger the number of distinct durations, the flatter the histogram will tend to be, other things being equal. If the distances in the multiset are spread out over the histogram bins, the heights of the histogram towers will tend to decrease. Indeed, for the six 16-pulse clave patterns of Figure 6, the lowest number of distinct durations is four, realized by the *Shiko* and the *Bossa-Nova*, both of which are almost regular, as can be seen more clearly in Figure 11.

When studying the *number of distinct* durations in a rhythm, the disparity between the geodesic distance between two points on a circle, and the chord length between the corresponding two points vanishes, since two chords have the same length if and only if their corresponding geodesic distances along the circle are equal. Therefore all the results in the mathematics literature that are concerned with distinct distances between vertices of convex polygons speak directly to the inter-onset duration analysis of rhythms, chords. and scales [135], [4], [5], [70], [71], [72].

Consider for example $v^{conv}(k)$, the minimum number of distinct distances among k points in convex position in the plane. In 1946, Paul Erdős [62] conjectured that for $k \geq 3$, $v^{conv}(k) = \lfloor k/2 \rfloor$. In 1952, Leo Moser [135] showed that $v^{conv}(k) \geq \lfloor (k+2)/3 \rfloor$. Since then Altman [4], [5] solved the problem by showing that $v^{conv}(k) = \lfloor k/2 \rfloor$, with equality if and only if the implied polygon is regular. Regular polygons are maximally even, as shown by Fejes Tóth [177]. Therefore, a low value of the number of distinct durations in a rhythm may be considered as a possible indicator of its evenness, at least for suitably large values of k. For low values of k counterintuitive examples exist. For instance, for $k = 3$ and $n = 12$ the rhythms A= [x x x . . . .] and B=[x . . . x . . x . . . .], have two and three distinct durations, respectively, and yet B appears to be the more even of the two. It is an open problem to determine the relationship between the evenness of a rhythm and the number of distinct durations it contains, as a function of the relative cardinalities of k and n.

In 1995, Peter Fishburn [70] identified all convex k-gons for even k that have exactly the minimum of $k/2$ intervertex distances. Also, for $k = \{3, 5, 7\}$ he identified all convex k-gons that have exactly $(k+1)/2$ intervertex distances, one more than the minimum. Fishburn's results identify an interesting family of extreme polygons. It turns out that for small values of k each of his polygons in this family corresponds to a rhythm timeline used in traditional world music for some value of n. Some notable examples in this family are listed in the following, where each polygon (rhythm) is identified with three notations: Fishburn's notation, box-notation, and interval vector, respectively. In Fishburn's notation the polygon $R_n - m$ denotes, in our context, a rhythm with n pulses and $(n - m) = k$ onsets.

1. $R_5 - 1$ = [x x x x .] = (1112) is the rhythmic pattern of the *Mirena* rhythm of Greece [88]. When started on the fourth onset, as in [x . x x x] it is the *Tik* rhythm of Greece [88].

2. $R_6 - 1$ = [x x x x x .] = (11112) yields the *York-Samai* pattern, a popular Arab rhythm [166]. It is also a handclapping rhythm used in the *Al Medēmi* songs of Oman [61].

3. $R_7 - 2$ = [x . x x . x x] = (21211) is the *Nawakhat* pattern, another popular Arab rhythm [166]. In Nubia it is called the *Al Noht* rhythm [88].

4. $R_7 - 1$ = [x x x x x x .] = (111112) is the rhythmic pattern of the *Póntakos* rhythm of Greece when started on the sixth (last) onset [88].

5. $R_8 - 1$ = [x x x x x x x .] = (1111112), when started on the seventh (last) onset, is a typical rhythm played on the *Bendir* (frame drum), and used in the accompaniment of songs of the *Tuareg* people of Libya [166].

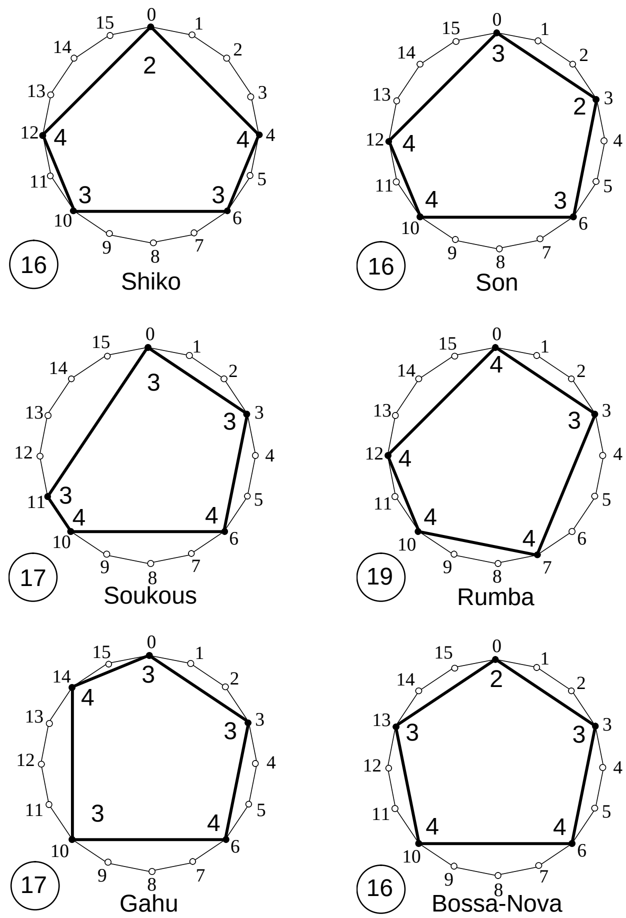


Figure 11: The number of distinct inter-onset durations for each onset is marked at each interior angle of the rhythm polygon. The diagonals are not drawn for the purpose of clarity.

6. $R_9 - 2 =$ [x . x x x . x x x] = (2112111) is the *Bazaragana* rhythmic pattern of Greece [88].

Also of interest in rhythm analysis is the importance of each onset to the overall rhythm. In particular, for a given onset, what influence does the number of its distinct inter-onset durations to all other onsets have on the salience of that onset? Figure 11 depicts the number of distinct inter-onset durations for each onset, for the six clave timelines.

This feature of convex polygons has also received attention from mathematicians. In 1975, Paul Erdős [63] conjectured that every set S of n points in convex position in the plane has one of its points p such that $dd_S(p)$, the number of distinct distances from p, is at least $\lfloor n/2 \rfloor$. For $n = 5$ the conjecture yields a value of 2. From Figure 11 we see that only the *Shiko*, the *Son*, and *Bossa-Nova*, which have an axis of symmetry passing through one of its onsets, match Erdős' conjectured lower bound of 2.

Another interesting feature of rhythms is $DD(S)$, the sum of the $dd_S(p)$ over all vertices of the polygon, indicated in Figure 11 by the number in the circle on the lower left corner of each rhythm pictured. This quantity has also been studied by mathematicians [70], [71], [72]. An onset that has many distinct distances to the other onsets in a rhythm may be considered rich and complex in

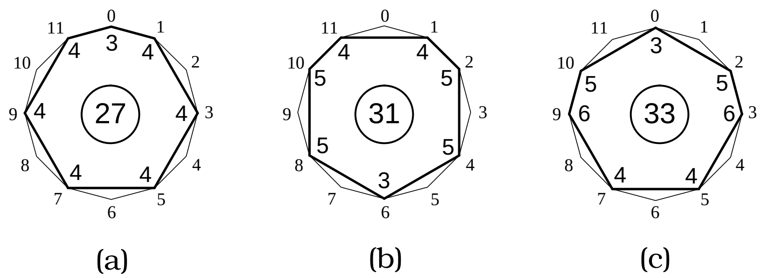


Figure 12: The three necklace patterns of the seven-onset 12-pulse bell rhythms.

some sense. Therefore a rhythm with a large value of $DD(S)$ has an overall richness, at least from the mathematical point of view. The *Rumba* is considered to be quite special from the musicological and geometric points of view [183]. Its polygon has no right-angle vertices, no axes of symmetry, and no two equal adjacent inter-onset durations. It is interesting to note here that its $DD(S)$ value is 19, the highest value in Figure 11, thus providing additional mathematical evidence of the rhythm's uniqueness. It would be interesting to determine if this mathematical uniqueness has musicological or psychological explanatory relevance. From the mathematical point of view it would be interesting to determine what are the relationships, if any, between the $DD(S)$ value of a rhythm and its evenness.

As a second example consider the African Sub-Saharan bell patterns that contain seven onsets in a time span of twelve units [185]. One feature that these patterns have in common is that the adjacent inter-onset duration intervals come in only two sizes: one and two. Under this restriction there are only 21 possible rhythms that begin on an onset. Of these 21 only 11 are used in the traditional music of this part of Africa. In addition, there are only three possible rhythm necklaces. These three necklaces are shown in Figure 12. In different parts of Africa different names are used for the 11 rhythms depending on which necklace is used and on which onset the rhythm is started. The necklace of Figure 12 (a) yields only one documented rhythm, and it starts on position 11. The necklace of Figure 12 (b) gives rise to three documented rhythms which start on positions 2, 4, and 8. On the other hand, the necklace of Figure 12 (c) determines seven rhythms: it is started on all seven of its onsets. There is clearly a preference relation here: (b) is preferred over (a), and (c) is much prefered over the other two. An analysis of these three necklace patterns from the point of view of the number of distinct durations suggests an open problem. The values of the function $DD(S)$ for the three necklaces in Figure 12 (a), (b), and (c), shown in the circles in the center of each polygon, are, respectively, 27, 31, and 33, suggesting that a high value of $DD(S)$ is desirable. Does this mathematical property have musical or psychological explanatory value? This suggests that a pattern with a high value of $DD(S)$ has rhythmic salience. The most preferred necklace of Figure 12 (c) has the additional interesting feature that it is the only one that has an onset p (in fact two of them diametrically apart) with a $dd_S(p)$ value of six. These two onsets, at positions 3 and 9, each have distinct durations to all other onsets.

4 Measuring the Similarity of Rhythms

At the heart of any algorithm for comparing, recognizing or classifying rhythms, lies a measure of the similarity between a pair of rhythms. The type of similarity measure chosen is in part predetermined by the manner in which the rhythm is represented. Furthermore, the design of a measure of similarity is guided by at least two fundamental ideas: *what* should be measured, and *how* should it be measured. The preceding sections discuss a variety of geometric features for representing rhythms. Additional geometric features may be found in [180], [192], [83], and [171]. Other important features of rhythm that may be used to compare rhythms include the amount of *syncopation* present in the rhythm [84]. Features traditionally used for measuring the similarity of musical chords and scales [152] may also be used for rhythm. In addition researchers in information retrieval have used a barrage of statistical features based on information theory [129], and on the inter-onset interval histograms [85]. Using d such features, a rhythm maps to a point in a d-dimensional feature space. In this setting the similarity between two rhythms may be calculated using any distance measure between their corresponding points in feature space. Then the entire arsenal of instance-based learning and data-mining tools may be brought to bear on the problems of rhythm analysis, classification, and retrieval from data bases [190].

A different approach views rhythms as sequences of symbols. There exists a wide variety of methods for measuring the similarity of two rhythms directly from such strings of symbols [184]. Indeed the resulting approximate pattern matching problem is a classical problem in pattern recognition and computer science in general [56]. Traditionally the similarity between two pattern strings is measured by a simple template matching operation, such as the Hamming distance, or (more recently) different variants of the *edit* distance. The Hamming distance between two equal-length strings of symbols is defined as the number of places in the strings where the corresponding symbols differ. Early versions of the edit distance used three operations: deletion, insertion, and replacement. Mongeau and Sankoff [130] extended the edit distance by adding the operations: *consolidation* and *fragmentation* in their study. In consolidation multiple notes are combined to form a single note. In fragmentation one note is segmented into multiple notes. A similar approach has been taken in computational phonology where these operations are called *compression* and *expansion*, respectively [114]. Hu and Dannenberg [98] showed experimentally that adding these two operations improves the quality of retrieval from sung queries. More recently similarity has been measured with more powerful and complex functions such as the *earth mover's distance* [29], [197], [202], the *proportional transportation distance* [79], weighted geometric matching functions [2], [125], the swap-distance [186], the directed swap-distance [54], [43], and the many-to-many minimum-cost matching distance [44], [41].

4.1 The swap-distance

The Hamming distance between two n-bit binary sequences is attractive from the algorithmic point of view because it may be trivially computed in $O(n)$ time. However, this distance is not appropriate for measuring rhythm dissimilarity, when used with a binary-string representation of rhythms, because it does not measure how far the mismatch between the two corresponding note onsets occurs. Furthermore, if a note onset is displaced a large distance, the resulting modified rhythm will in general sound considerably different from the original, and the Hamming distance may not be sensitive to such changes. To combat this inherent weakness of the Hamming distance, variants and generalizations have been proposed over the years. One early generalization is the *edit* distance which allows for insertions and deletions of onsets. Discussions of the application of the *edit-distance* to the measurement of similarity in music can be found in Mongeau and Sankoff [130] and Orpen and Huron [138]. A noteworthy more recent generalization is the *fuzzy* Hamming distance [19] which allows *shifting* of onsets as well as insertions and deletions. Using *dynamic programming* these distances may be computed in $O(n^2)$ time in the worst case. Bookstein et al. [19] gave an algorithm for computing the fuzzy Hamming distance in $O(n + k_1 k_2)$ time, where n is the number of pulses in the rhythms, and k_1 and k_2 are the numbers of onsets in each rhythm. Minghui Jiang improved this complexity to $O(n)$ [103].

The problem of comparing two binary strings of the same length with the same number of one's suggests an extremely simple edit operation called a *swap*. A swap is an interchange of a one and a zero that are adjacent to each other in the binary string. Interchanging the position of elements in strings of numbers is a fundamental operation in many sorting algorithms [49]. However, in the sorting literature a swap may interchange non-adjacent elements, and is also called a *transposition*. The *transposition-distance* (also called Cayley distance) between two sequences is the minimum number of transpositions needed to convert one sequence to the other. When the elements are required to be adjacent, the swap has been called a *mini-swap* or *primitive-swap* [15], as well as *adjacent-swap* [132]. In computational biology a related operation called a *short-swap* is also of interest, in which two elements are switched if they have at most one element betwen them. The *short-swap* distance is the minimum number of short-swaps required to convert one sequence to another. Heath and Vergara [94] give an algorithm that computes an approximation of the short-swap distance in $O(n^2)$ time that is within twice the optimal value. Here we use the term *swap* to mean the interchange of two adjacent elements. The swap-distance between two rhythms is the *minimum* number of swaps required to convert one rhythm to the other. The swap-distance may be viewed as a simplified version of the *generalized* Hamming distance [19], where only the shift operation is used, and the cost of the shift is equal to its length. It has also been used in non-parametric statistics to compare two sequences in the context of rank-correlation, and corresponds to *Kendall's* τ [110], [131]. When one sequence is a perfectly ordered sequence it can be used as a measure of disarray, as done by Diaconis and Graham [53], who determine several relations between the swap-distance and other metrics on the set of permutations of sequences.

The swap distance is more appropriate than the Hamming distance in the context of rhythm similarity [183], [185]. It is also a special case of the more general *earth mover's distance* (also called *transportation distance*) used by Typke et al. [197] to measure melodic similarity. Given two sets of points called supply points and demand points, each assigned a weight of material, the earth mover's distance measures the minimum amount of work (weight times distance) required to transport material from the supply points to the demand points. No supply point can supply more weight than it has and no demand point receives more weight than it needs. Typke et al. [197] solve this problem using linear programming, a relatively costly computational method. In particular the simplex algorithm could take an exponential number of steps, and the polynomial complexity interior-point methods are not as fast as the methods described in the following. The swap-distance is a one dimensional version of the earth mover's distance with all weights equal to one. Furthermore, in the case where both binary sequences have the same number of one's (onsets), there is a one-to-one correspondence between the indices of the ordered onsets of the sequences [108].

The swap-distance may of course be computed by actually performing the swaps, but this is inefficient. If X has one's in the first $n/2$ positions and zero's elsewhere, and if Y has one's in the last $n/2$ positions and zero's elsewhere, then a quadratic number of swaps would be required. On the other hand, if we compare distances instead, a much more efficient algorithm results. First scan the binary sequence and store a vector of the x-coordinates at which the k onsets occur (the *onset-coordinate vector*). Then the swap-distance between the two onset-coordinate vectors U and V with k onsets may be computed with the following formula:

$$d_{SWAP}(U,V) = \sum_{i=1}^{k} |u_i - v_i|, \tag{1}$$

which is the L_1 norm of the vector $U - V$. This approach has also been applied to measure chord similarity in the context of *voice-leading* [194], [195], [196]. The set of k distances $|u_i - v_i|$ are called the *displacement multiset* in the theory of voice-leading [195], [90]. Note that the swap-distance implies a *one-to-one* mapping (or perfect matching [80]) between the onsets of U and those of V. Tymoczko [196] calls this a *bijective* mapping. Computing U and V from X and Y is done trivially in $O(n)$ time with a simple scan. Therefore $O(n)$ time suffices to compute $d_{SWAP}(U,V)$, resulting in a large gain over using linear or dynamic programming. For a survey of metrics on permutations see [52].

Whenever it is desired to measure the distance between two objects one invariably must make a choice about what metric to use: the L_1, L_2, L_∞, some other norm such as the L_p norm (p-Minkowski metric), or any of scores of other possibilities [147]. The answer invariably depends on the application: is computational complexity important, do we want good performance out of a machine [179], do we want mathematical tractability, or do we want to faithfully model human perception? The swap-distance, measures the distance (duration) between onsets, and so it naturally leads to the L_1 norm. One could of course use the L_2 norm, which would assign more weight to longer durations. The L_1 norm (*taxi-cab* metric) is popular as a measure of *voice-leading distance* [39], [119], [156], [168]. Michael Keith [109] argues that it is a more musically important metric for comparing scales because it is a good measure of their perceptual closeness, especially when the distances are small. On the other hand, Clifton Callender [26] uses the L_2 norm in his research because it is more tractable from the algebraic point of view.

4.2 The directed swap-distance

The swap-distance between two rhythms makes sense only if both rhythms have the same number k of onsets. In a more general setting, the two rhythms have different values of k, and the algorithm described in the preceding is not well defined. In order to capture the attributes of the swap-distance for rhythms with unequal numbers of onsets we may use the *directed* swap-distance, first applied successfully to the phylogenetic analysis of flamenco metric rhythms [54], [55], and more recently to the analysis of Steve Reich's *Clapping Music* and the *Yoruba* bell timeline [42]. The directed swap-distance is defined as the minimum number of swaps required to move every element of S to the index (position) of an element of T, with the restriction that every element of T must have at least one element of S moved to its index. This mathematical measure of similarity is intuitively satisfying, is used in bioinformatics to compare molecular sequences, and when it was recently applied to the phylogenetic analysis of flamenco metric rhythms, it confirmed several beliefs that musicologists have about the evolution of flamenco music [54], [55]. Furthermore, experiments with human subjects on the same metric rhythms yielded dissimilarity matrices and phylogenetic trees with the same structure, thus confirming that the directed swap-distance can model human judgements of rhythm dissimilarity [1].

The directed swap-distance may be viewed as a linear *assignment* problem [108], where the cost of an assignment between an element i of S and an element j of T is the distance between i and j. Furthermore, we may consider the more general input consisting of two sets of unsorted *real* numbers on the *real* line rather than binary sequences. Here the real numbers play the role of the indices of the one's in the binary sequence (except that in a binary sequence the one's are already sorted). In this setting, if both sets have equal cardinalities, the simple algorithm described in the

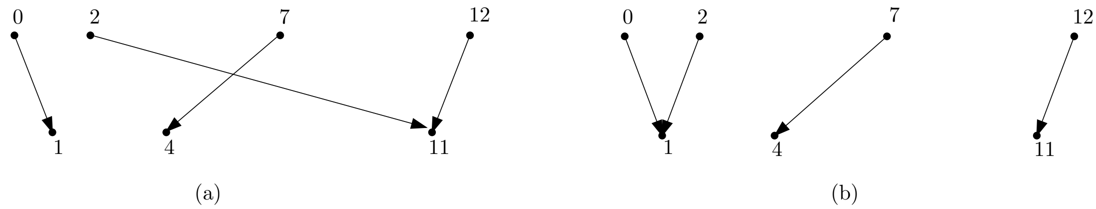


Figure 13: (a) A surjection between sets $S = \{0,\ 2,\ 7,\ 12\}$ and $T = \{1,\ 4,\ 11\}$. (b) A minimal surjection between S and T.

preceding for binary sequences may still be used after sorting the real numbers, thus yielding an $O(n \log n)$ time algorithm.

An alternate way of viewing the directed swap-distance is as a surjection, ψ, between two sets of elements S (the source) and T (the target) on the interval $(0, X)$ where $|S| \geq |T|$. This mapping is bound by the constraint that each element of T must have at least one element of S mapped to it. More formally the directed swap-distance may be expressed as a surjection as follows:

$$\min_{\psi} \sum_{s \in S} |s - \psi(s)|. \tag{2}$$

Any surjection that satisfies the preceding equation is a minimal surjection. Figure 13 depicts two different surjections between two sets of points on the line, one of which is minimal. Note that all the points actually have zero y-coordinates; they are shown in this way merely for the purpose of clarity.

In 1979 the philosopher Graham Oddie proposed using surjections to measure the distance between two theories expressed in a logical language [136]. In 1997 Eiter and Mannila extended this idea by expressing theories as models, and thus as points in a metric space [59]. This gave them a new distance measure in a metric space which they called the surjection distance. The surjection distance between two sets S and T is defined as follows:

$$\min_{\psi} \sum_{s \in S} \delta(s, \psi(s)), \tag{3}$$

where δ is a distance metric on the space, and ψ is a surjection between S and T. They also proposed an algorithm for computing the surjection distance in $O(n^3)$ time, where $n = |S|$, by reducing the problem to finding a minimum-cost perfect matching in an appropriate graph [80].

In 2003, Ben-Dor et al. [14], in the context of the shotgun sequencing problem in computational biology, introduced an assignment problem similar to the directed swap-problem where the points are real numbers on the line rather than one's in a binary string: the *restriction scaffold assignment* problem. They also presented an $O(n \log n)$ algorithm to compute this assignment problem. Their result relies heavily on a result of Karp and Li [108] which provides a linear time algorithm (after sorting) for computing the *one-to-one* assignment problem in the special case where all the points lie on a line. Of course, in the one-to-one assignment problem between S and T some elements of S remain unassigned. Colannino and Toussaint [43] give a counter-example to the algorithm of Ben-Dor et al., [14], and show that the problem may be solved in $O(n^2)$ time, an improvement over the previous best algorithm with running time $O(n^3)$ [59]. Colannino et al., [40] further improved this complexity to $O(n \log n)$ for real points on the line, and to $O(n)$ for rhythms expressed as binary sequences.

4.3 The many-to-many matching distance

Although the directed swap-distance between two rhythms (or equivalently, the minimal surjective voice-leading between two chords, in Tymoczko's terminology [194], [195], [27]) gave good results in the case of flamenco metric rhythms [54], [55], this measure suffers from some drawbacks, in general. For example, given two rhythms such as A=[x . x . . . . x .] and B=[. x . . . . x . x], the directed swap-distance is realized by assigning the first, second, and third onsets of A to the first, second, and third onsets, respectively, of B, to give a distance of 9. On the other hand, a more satisfying assignment would assign the first and second onsets of A to the first onset of B, and the second and third onsets of B to the third onset of A for a total distance of 4. In other words onsets should be able to split or merge in both directions. There are several ways in which the directed swap-distance may be generalized so as to handle these *fusion* and *fission* operations. One approach is to compute minimum-cost *many-to-many* matchings between the two sets of real numbers that represent the time points of the two rhythms in question, as was done in [41]. For example, if we let S and T denote the two sets of points with total cardinality n, the minimum-cost many-to-many matching problem matches each point in S to at least one point in T and each point in T to at least one point in S, such that sum of the matching costs is minimized, where both S and T lie on the line, and the cost of matching $s \in S$ to $t \in T$ is equal to the distance (or L_1 norm) between s and t. In this context, [41] provides an algorithm that determines a minimum-cost many-to-many matching in $O(n \log n)$ time, improving the previous best time complexity of $O(n^2)$ for the same problem [44]. In music theory this minimum-cost *many-to-many* matching is called the minimum voice-leading for arbitrary chords, when the points lie on a circle and the minimum over all rotations is desired. Tymoczko [195] gives a dynamic programming algorithm for computing a solution in $O(n^3)$ time for two chords of n notes.

It is worth pointing out that the continuous version of this mapping has many applications in music, such as the evaluation of *beat-trackers* and *metrical models* [176], continuous voice-leading [26], as well as *score-performance* matching [95].

5 Adding Pitch to Rhythm

Just as rhythm may be represented as a one-dimensional onset function of time, melody may be considered as a two-dimensional onset function of time and pitch, where each onset is given a pitch value. A melody may then be represented as a Manhattan skyline [91] (also called the *piano-roll* representation [175]). In fact, the well-known composer Heitor Villa-Lobos composed pieces based on the New York City skyline, as well as the upper envelope contour of the mountains surrounding the city of Rio de Janeiro [91].

5.1 Measuring melodic similarity

A good introduction to the various approaches used for measuring melodic similarity may be found in reference [96]. Here we restrict ourselves to recent geometric approaches. ÓMaidín [137] proposed a geometric measure of the distance between two melodies modelled as x-monotonic pitch-duration rectilinear functions of time as depicted in Fig. 14. This is equivalent to representing notes as line segments in a pitch-time space when a note does not end before another begins [198]. ÓMaidín measures the distance between the two melodies by the area between the two resulting polygonal chains (shown shaded in Fig. 14). If the area under each melody contour is equal to *one*, the functions can be viewed as probability distributions, and in this case ÓMaidín's measure is identical to the classical Kolmogorov *variational distance* used to measure the difference between two probability

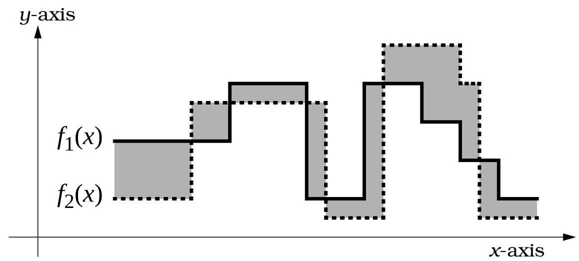


Figure 14: Two melodies as rectilinear pitch-duration functions of time.

distributions [182]. Polansky [147], who provides an exhaustive survey of metrics for music applications, calls this the *magnitude metric*. Note that if it is desired to measure the joint similarity of a group of melodies, a natural generalization of ÓMaidín's measure is Matusita's measure of *affinity* [181]. If the number of vertices (vertical and horizontal segments) of the two polygonal chains is n then it is trivial to compute ÓMaidín's distance in $O(n)$ time using a line-sweep algorithm.

In a more general setting, such as music information retrieval systems, we are given a short query segment of music, denoted by the polygonal chain $Q = (q_1, q_2, ..., q_m)$, and a longer stored segment $S = (s_1, s_2, ..., s_n)$, where $m < n$. Furthermore, the query segment may be presented in a different *key* (transposed in the vertical direction) and in a different *tempo* (scaled linearly in the horizontal direction). Note that the number of keys (horizontal levels) is a small finite constant. Time is also quantized into fixed intervals (such as eighth or sixteenth notes). In this context it is desired to compute the minimum area between the two contours under vertical translations and horizontal scaling of the query. Francu and Nevill-Manning [75] claim that this distance measure can be computed in $O(mn)$ time but they do not describe their algorithm in detail. In the more general setting where the two melodies are represented by two monotonic orthogonal chains with m and n vertices, and it is desired to compute the minimum area between the two curves, under vertical and horizontal translations, the problem comes up in a computer vision problem of matching polygonal shapes. Arkin et al. [9] show that this minimum area function is a metric, and that it can be computed in $O(n^3)$ time. Aloupis et al. [2], [3] improved this complexity to $O(nm \log(n + m))$ time.

5.2 Measuring chord similarity

There is a large literature in music theory that deals with the problem of measuring the similarity of chords [116], [174], [133], [124], [152], [102], [157]. Eric Isaacson [102] provides an in-depth discussion of many of these measures. The more traditional measures tend to assess, in one way or another, the number of pitches that the two chords have in common [155], [152]. Some measures assess the similarity of the *adjacency interval vectors*, such as Roeder [156], Chrisman [33], [34], and Regener [155]. Many measures are based on comparing the interval vectors (histograms) of the two chords. For example, Teitelbaum [174] computes the Euclidean distance between the two interval vectors, whereas Lord [124] and Rahn [152] calculate the city-block distance (also Manhattan metric or L_1 norm). However, comparing two chords (or two rhythms for that matter) by measuring the similarity between their interval vectors, disregards the fact that the two patterns may be quite different structurally, as the examples in Figures 7 and 10 illustrate. Indeed, psychological studies have shown that chords with four or more notes may sound quite differently from each other, even though they may have exactly the same interval vector [107]. In addition to the psychological aspects of intervallic perception, the physical (acoustic) aspects also play a role [18]. Furthermore, the

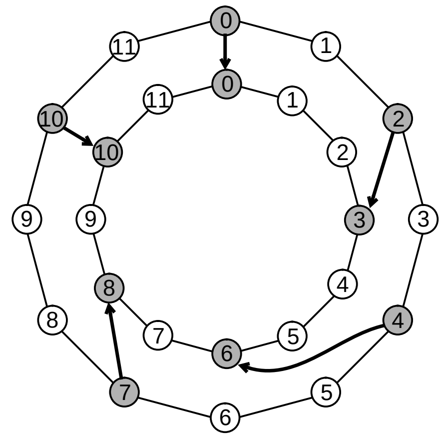


Figure 15: A voice-leading from a *source* chord S (outer circle) to a *target* chord T (inner circle).

preceding measures must be distinguished from the composer-oriented cognitive models of musical distance [27].

The approaches taken recently to measure rhythm-dissimilarity, discussed in the preceding section, based on the concept of an assignment, such as the swap-distance [183], [185], the directed-swap-distance [54], [55], [43], and the many-to-many matching distance [44], [41] may turn out to be quite useful for measuring chord similarity as well. Indeed such an approach has been suggested by Dmitri Tymoczko [194], [195], [196]. Determining how well these distance measures model human perception in both the time and pitch domains is a project under investigation [1].

5.3 Voice-leading

Consider two chords (or rhythms) such as S=[x . x . x . . x . x . .] and T=[x . . x . . x . x . x .], pictured in Figure 15, where S, the *source* chord is contained in the outer circle, and T, the *target* chord in the inner circle. In coordinate vector representation the chords are given by S=(0,2,4,7,10) and T=(0,3,6,8,10). A *voice-leading from S to T* is a function $V(S,T)$ which maps each element of S to an element of T [119]. For example, in Figure 15, S_0 maps to T_0, S_2 maps to T_3, S_4 maps to T_6, S_7 maps to T_8, and S_{10} maps to T_{10}, as indicated by the arrows. Each element in these chords constitutes a *voice*; it has it's own characteristic sound. Voice-leading functions $V(S,T)$ provide sets of rules that constrain the mapping in musically relevant ways. Lewin [119] discusses four types of voice-leading rules closely related to the methods of measuring rhythm similarity discussed in the preceding. In *maximally close voice-leading* each voice in S is mapped to its nearest voice in T. In *downshift voice-leading* each voice in S is mapped to its nearest counter-clockwise voice in T. In *upshift voice-leading* each voice in S is mapped to its nearest clockwise voice in T. The voice-leading shown in Figure 15 is an instance of an *upshift voice-leading*. These voice-leadings are similar in spirit to the algorithms for binarization and ternarization of rhythms [81], [82]. Finally, in a *maximally uniform voice-leading* S differs as little as possible from any transposition of T. This concept is analogous to the *necklace swap-distance*, i.e., the swap-distance between two rhythms, minimized over all possible rotation alignments between the two rhythms [187], [191], [8], [22], [37].

One important property of effective voice-leadings is the *no crossing principle* [99], [90], in which one "edge" of $V(S,T)$ should not properly cross another. For example, mapping S_0 to T_3 and S_2 to T_0 would produce such a crossing. Music theorists exploring voice-leading proceed from such general principles of voice-leading to arrive at salient definitions of distances between chords. On the other hand, in [44], [43], and [40] distance measures are defined intuitively, their mathematical properties investigated, and then they are tested pychologically. For example, it is shown in [44], [43], and [40], that the swap-distance, directed swap-distance, and the minimum-cost many-to-many matchings between two rhythms (scales, chords, or voice-leadings) yield no crossings.

One of the main desired effects of voice-leading rules is to make sure that different melodies (or rhythms) are heard separately while they are being played simultaneously. This property is called *streaming* by Bregman [21], who has characterized streaming as a competition between possible alternative cognitive organizations. Streaming is not simply a matter of relative proximity of two successive pitches in order to form a stream. The pitches must be closer than other possible pitch-time traces. David Huron provides a detailed discussion of voice-leading rules [99]. Robert Morris [134] gives a taxonomy of voice-leading types and a list of nine *condition sets*. For additional key papers on voice-leading the reader is referred to [119], [134], [168], [194], [195], [196].

6 Conclusion and Open Problems

In this section we list additional open problems that complement those scattered throughout the preceding sections. Let us assume that we are given a circular lattice with n points (evenly spaced), and we would like to create a rhythm consisting of k onsets by choosing k of these n lattice points. For example, perhaps $n = 16$ and $k = 5$ as in Figure 1. Furthermore we would like to select the k onsets that maximize the sum of the lengths of all pairwise chords (according to some measure) between these onsets. Evaluating all n-choose-k subsets may in general be too costly. However, interesting rhythms often have additional musicological constraints that may be couched in a geometric setting [10], [30], [185], [188], [192]. These properties may permit simpler solutions than brute force methods. The special case of maximizing the sum of the pairwise distances suggests a general approximation method with the following *snap* heuristic: construct a regular k-gon inscribed in the circle, and then move its vertices to their nearest points on the n-lattice. For definiteness, if a vertex of the regular polygon is equidistant to two points of the n-lattice, move it to the nearest point in a clockwise direction. One would expect such a rhythm to have a high evenness value under most reasonable definitions of evenness. Indeed, for this case it is known that this snap heuristic yields maximally even rhythms [51], [38], [17]. How close to optimal is this procedure according to other known measures of evenness? Also of interest is computing the sum of all the pairwise distances efficiently. Minghui Jiang [104] showed that the snap heuristic finds the optimal solution when the measure is the sum of the pairwise arc-lengths, and gives an $O(k)$ time algorithm for computing the resulting sum, when the k points are given in sorted order by polar coordinates.

The two sequences shown in Figure 7 are the only possible rhythm bracelets with flat histograms, for any values of k greater than three [152]. Therefore in order to be able to generate additional rhythms the above constraints need to be relaxed. We may proceed in several directions. For example, it is desirable for timelines that can be played fast, and that "roll along" (such as the *Gahu* already discussed), that the rhythm contain silent gaps that are neither too short nor too long. Therefore it would be desirable to be able to efficiently generate rhythms that either contain completely prescribed histogram shapes (interval vectors), or have geometric constraints on their shapes, and to find good approximations when such rhythms do not exist. One may also ask for rhythms with prescribed distinct-distance vectors. This area of research is almost unexplored. A notable exception is the work of Nicholas Collins [47] who explores the effects of the existence of high histogram columns (distances in the interval vector with high multiplicities) on the uniqueness of the rhythms that have a given interval vector.

The preceding discussion on the swap-distance was restricted to comparing two linear strings.

However, many rhythms (the timelines in particular) are cyclic, and there are applications in (music information retrieval) in which it is desired to compute the best alignment of two cyclic rhythms over all possible rotations. In other words, it is of interest to compute the distance (according to some appropriate measure that depends on the application) between two rhythms, minimized over all possible rotations of one with respect to the other. The same problem is of interest in voice-leading [196]. Some work has been done with cyclic string matching for several definitions of string similarity [86], [120], [126], [35]. Consider two binary sequences of length n and density k (k ones and $(n-k)$ zeros). It is desired to compute the minimum swap-distance between the two strings under all possible alignments. I call this distance the *cyclic swap-distance* or also the *necklace swap-distance*, since it is the swap-distance between two necklaces. From the preceding discussion it follows that the cyclic swap-distance may be computed in $O(n^2)$ time by using the linear-time algorithm in each of the n possible alignment positions of the two rhythms. Note that swaps may be performed in whatever direction (clockwise or counter-clockwise) yields the fewest swaps. In 2002 I asked whether the cyclic swap-distance may be computed in $o(n^2)$ time? In contrast, if the swap-distance is replaced with the Hamming distance, then the cyclic (or necklace) Hamming distance may be computed in $O(n \log n)$ time with the Fast Fourier Transform [73], [87]. Since I posed this problem, Jeff Erickson pointed out that the necklace swap-distance problem can be transformed into a problem known as the minimum-convolution problem. The obvious minimum-convolution algorithm runs in $O(n^2)$ time. A similar algorithm solves the analogous maximum-convolution problem. Bussieck et al. [25] describe an algorithm that runs in $O(n \log n)$ expected time if the input arrays are randomly permuted, but still runs in $O(n^2)$ time in the worst case. Ardila et al. [8] show that the cyclic swap-distance may be computed in $O(n + k^2)$ time where k is the number of one's in the sequence. Of course in the case of rhythms, $k = O(n)$ and thus this complexity is still $O(n^2)$. See also the work by Clifford and Iliopoulos [37]. Bussieck et al. [25] also asked whether $o(n^2)$ time was possible. This question was finally answered affirmatively by Timothy Chan. The necklace swap-distance problem asks for the optimal rotation of two given necklaces of n beads at arbitrary positions to best align the beads in the sense that the resulting number of swaps is minimized. This in effect asks for the optimal rotation (necklace alignment) that minimizes the L_1 norm between the positions of the beads. In a more general setting we can ask the same question for any L_p norm. Bremner et al., [22] obtain solutions for $p = 1, 2, \infty$. In particular they show that in the standard real RAM model of computation the L_1 necklace alignment may be solved in time $O(n^2(\lg \lg n)^2 / \lg n)$, the L_2 necklace alignment may be solved in time $O(n \lg n)$, and the L_∞ necklace alignment may be solved in time $O(n^2 / \lg n)$.

The work of ÓMaidín [137] and Francu and Nevill-Manning [75] suggests several interesting open problems. In the acoustic signal domain the key of the melody loses significance and hence the vertical transposition is continuous rather than discrete. The same can be said for the time axis. What is the complexity of computing the minimum area between a query $Q = (q_1, q_2, ..., q_m)$ and a longer stored segment $S = (s_1, s_2, ..., s_n)$ under these more general conditions?

A simpler variant of the melody similarity problem concerns acoustic *rhythmic* melodies, i.e., cyclic rhythms with notes that have pitch as a continuous variable. Here we assume two rhythmic melodies of the same length are to be compared. Since the melodies are cyclic rhythms they can be represented as closed curves on the surface of a cylinder. What is the complexity of computing the minimum area between the two rectilinear polygonal chains under rotations around the cylinder and translations along the length of the cylinder? Aloupis et al. [2] present an $O(n)$ time algorithm to compute this measure if rotations are not allowed, and an $O(n^2 \log n)$ time algorithm for unrestricted motions (rotations around the cylinder and translations along the length of the cylinder). It turns out that this problem is identical to a computer vision problem of matching polygonal shapes, for which Arkin et al. [9] give an $O(n^3)$ time algorithm. Can the $O(n^2 \log n)$ time be improved?

In the preceding sections several tools were pointed out that can be used for computer composition. We close the paper by mentioning one additional tool for automatically selecting good rhythm timelines. In [189] it is shown that the *Euclidean* algorithm for finding the greatest common divisor of two numbers can be used to generate interesting rhythm timelines when the two numbers that serve as input to the *Euclidean* algorithm are the number of onsets (k) and the time-span (n), respectively, of the desired rhythm. The resulting rhythms are particularly attractive when k and n are relatively prime [171], [51]. Indeed, this algorithm generates a large fraction of all timelines used in world music, with the notable exception of Indian *talas* [36].

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