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though a finite collection of star-shaped polygons can interlock, in the sense that it may be possible to translate one polygon to ∞ without disturbing the others, the collection can nevertheless be separated by *simultaneous translations*. It is an open problem whether this is true for monotone polygons.

Similar problems in three dimensions are being explored. The definition of star-shaped polygons can be extended to three dimensions and the previous results of Dawson[8] apply there too. Thus two star-shaped polyhedra are separable with a single translation. The notion of monotonicity is not straightforwardly extended to three dimensions. One common definition of a monotone polygon is as follows: a polygon P is monotonic in direction θ if every line orthogonal to θ that intersects the interior of P yields a *line segment* as the intersection. Extending this definition to three dimensional space yields a family of polyhedra termed *weakly-monotonic* polyhedra,[20] where a polyhedron P is weakly monotonic in direction θ if every plane orthogonal to θ that intersects the interior of P yields a *simple polygon*. By restricting the class of simple polygons obtained in the intersection, we obtain subclasses of polyhedra. Note that, unlike star-shaped polyhedra, even *weakly-monotonic* polyhedra with *convex* intersections need not be separable under translations. One example is provided by constructing two such polyhedra in the form of the *double helix*. The only way to separate these monotonic polyhedra is with a *screw motion*, i.e. a *simultaneous* translation and rotation.

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single translation in direction $\varphi + \pi/2$.

This theorem immediately suggests the following algorithm for determining a direction of separability for two monotone polygons.

Algorithm SEPARATE

Input. Two non-intersecting monotone polygons P = $(p_1, p_2,..., p_n)$ and Q = $(q_1, q_2,..., q_n)$.

Output. A direction ψ for separating P and Q.

BEGIN

Step 1: Compute the directions of monotonicity for P and Q.

Step 2: If P and Q have a common direction of monotonicity ζ, then EXIT with ψ <-- $\zeta + \pi/2$.

Step 3: Pick two directions of monotonicity for P and Q, say θ and φ respectively.

Step 4: Compute VH(Q, $\theta + \pi/2$).

Step 5: If P intersects VH(Q, $\theta + \pi/2$)

then EXIT with ψ <-- $\varphi + \pi/2$

else EXIT with ψ <-- $\theta + \pi/2$

END

Theorem 3.2: Algorithm SEPARATE determines a direction of separability for two monotone polygons P and Q in O($n + m$) time.

Proof: The correctness of the algorithm follows from theorem 3.1. Thus we turn to complexity. Steps 1-3 can be performed in O($n + m$) time using the algorithm in ref. 15 by Preparata and Supowit. Computing the visibility hull of Q in step 4 can be done in O(m) time with a variety of hidden line algorithms.[16,17] Finally, step 5 can be performed in O($n + m$) time using a simple modification of the slab method of Shamos and Hoey[18] for intersecting two convex polygons. This follows from the fact that P and VH(Q, $\theta + \pi/2$) are two monotonic polygons in direction θ and therefore their intersection can only contain a linear number of pieces. See, for example, Guibas and Stolfi.[19]

4. Concluding Remarks

We have shown that two monotone polygons in the plane are movably separable with a single translation and that a direction for translation can be found in linear time. Thus monotone polygons and star-shaped polygons share this movable separability property. Dawson[8] has shown that al-

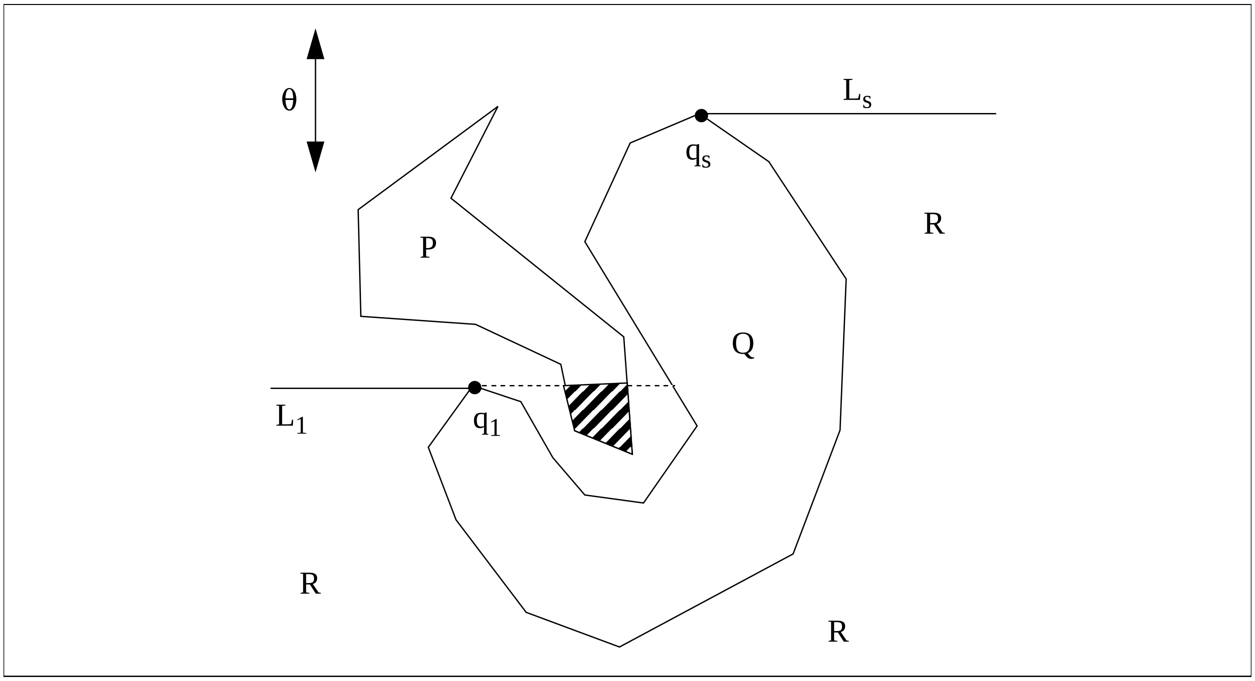

Fig. 9. Visibility Hull of Q.

Case 1: P does *not* intersect the interior of VH(Q, θ + π/2). In this case, since VH(Q, θ + π/2) is a polygon monotonic in direction θ, we have two polygons, the interiors of which do not intersect, with a common direction of monotonicity θ. Therefore, by lemma 3.1, they can be separated by a single translation in direction θ + π/2.

Case 2: P *does* intersect the interior of VH(Q, θ + π/2). By lemma 3.4, P can only intersect *one* pocket of VH(Q, θ + π/2). There are four types of pockets but we need consider only one. For if P intersects a *top-left* pocket of VH(Q, θ + π/2), by mirror-symmetry transformations of the plane about the x and y axes we obtain the three other types of pockets without changing either the monotonicity direction θ of P or the separability of P and Q. Therefore, assume that P intersects the interior of a *top-left* pocket of VH(Q, θ + π/2). (Refer to Figure 9.)

Let Q' denote the *top-left* pocket of Q in question. Let q_s denote the first vertex of Q encountered when we traverse bd(Q) in a clockwise manner starting at a vertex in Q' such that no other vertex of Q has y coordinate greater than q_s. Let q_l denote the first vertex of Q encountered when we traverse bd(Q) in a counterclockwise manner starting at a vertex Q' such that q_l lies on the lid of the pocket Q'. Then q_l and q_s partition Q into two polygonal chains $Q_{ls} = (q_l,..., q_s)$ and $Q_{sl} = (q_s,..., q_l)$. Let L_s denote the half-line in the +x direction emanating from q_s and let L_l be the half-line in the -x direction emanating from q_l. The half-lines L_l and L_s together with the chain Q_{sl} partition the plane into two unbounded regions. Let R be the region not containing int(Q). First we note that P cannot intersect int(R). This follows from arguments similar to those of lemma 3.4. Now consider the direction of monotonicity φ of Q. Q cannot be monotonic in the y direction since then VH(Q, θ + π/2) would not contain any pockets. It is easy to see that for any other direction φ, the shadow SH(Q, φ + π/2) always lies in R. Furthermore, since P cannot intersect int(R), it can never intersect the interior of SH(Q, φ + π/2). It follows from lemma 3.3 that in this case P and Q can be separated with a

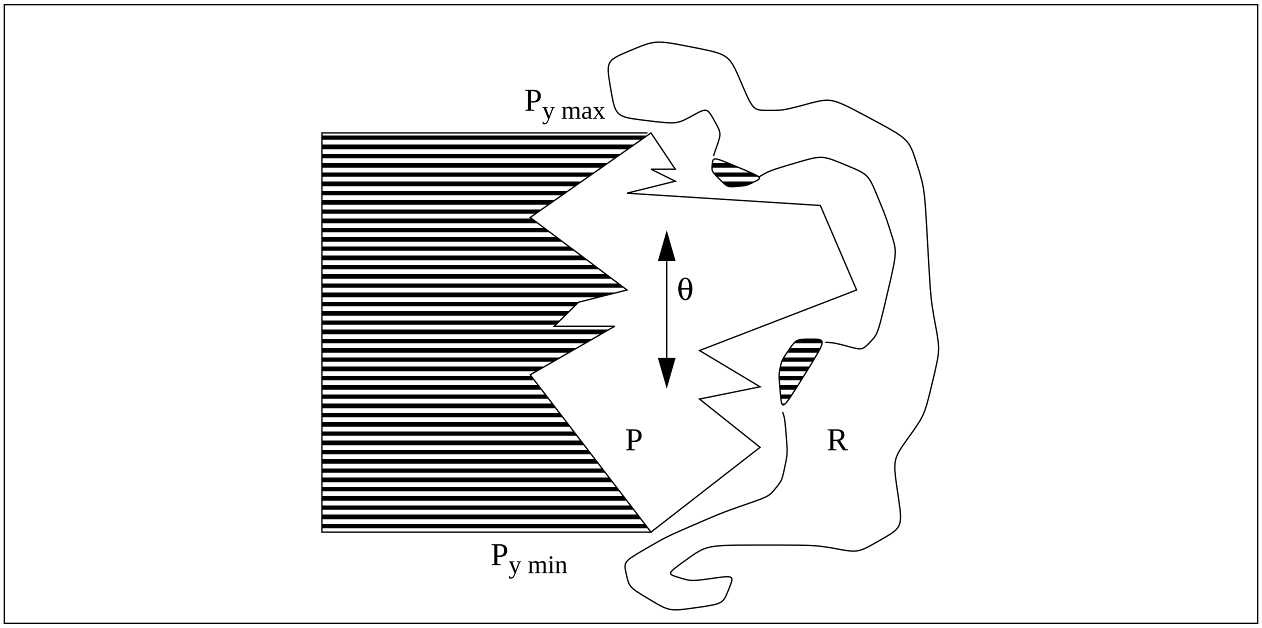

Fig. 8. Shadow of P in the -x direction.

Proof: Let θ be the y-axis and consider the shadow of P in the -x direction. (see Figure 8.) The vertices of P with maximum and minimum y coordinates (p_{ymax} and p_{ymin}) partition bd(P) into two polygonal chains: the *left* chain and the *right* chain. Since R does not intersect interior of SH(P, -x), it follows that the left chain of P can be translated in the -x direction. Now consider the convex hull pockets of P on the right chain. The intersection of R with CH(P) may yield a set of polygons R_1, R_2,..., R_r. Let R' denote the part of R not intersecting CH(P). Since R' intersects neither CH(P) nor SH(P, -x) it follows that R' can be translated in the +x direction. Furthermore, it follows from lemma 3.2 that each R_i, $i = 1, 2,..., r$ can also be translated in the +x direction. Therefore, R itself can be so translated. It follows that the right chain of P, and therefore P itself, can be translated in the -x direction without colliding with R.

Lemma 3.4: Let P and Q be two polygons monotonic in direction θ and φ, respectively. Then P cannot intersect the interior of more than *one* pocket of VH(Q, θ + π/2).

Proof: Let θ be the y-axis. Since P is monotonic in the direction of y-axis, the y coordinate of a point traveling the bd(P) can have at most two local extrema. From the definition of a pocket, it follows that if P intersect the interior of more than one pocket of VH(Q, θ + π/2) we will have more than two local extrema, a contradiction.

We are now ready to present the main result of the paper.

Theorem 3.1: Given two polygons P and Q monotonic in directions θ and φ, respectively, then P and Q are separable with a single translation in at least one of the two directions θ + π/2, φ + π/2.

Proof: Without loss of generality we assume θ to be the y axis. Now construct the visibility hull of Q in the x direction, VH(Q, θ + π/2). (Refer to Figure 9.)

Two cases arise:

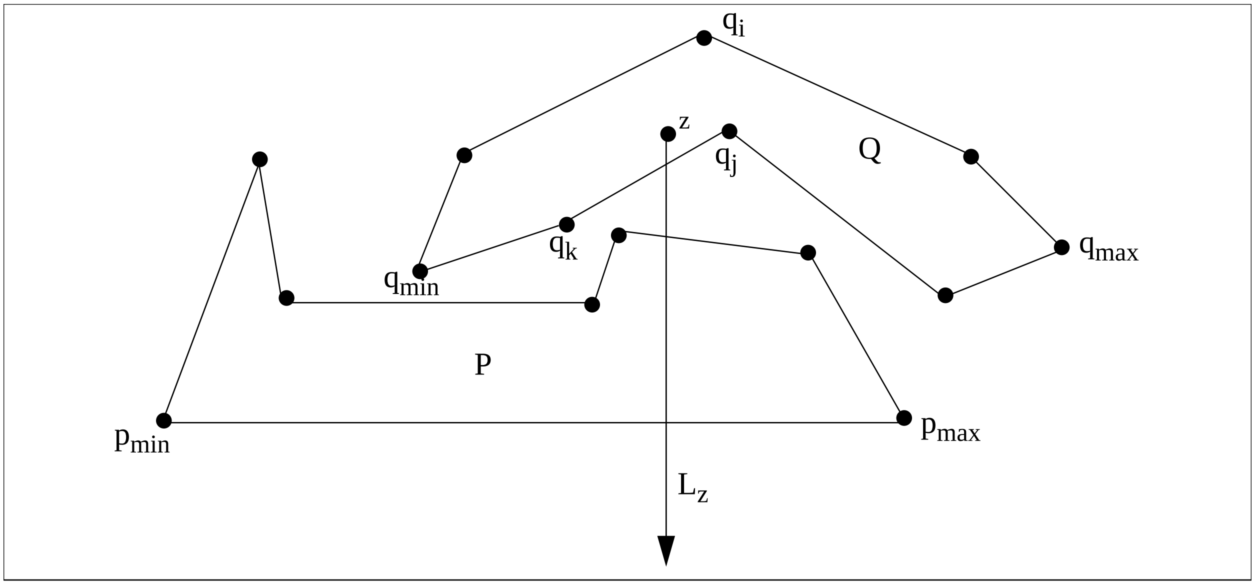

Fig. 6. Two polygons with θ as the x-axis.

Proof: Let P' be the convex hull pocket of P containing R. Without loss of generality, let θ be the direction of the y-axis and P' be a *right* pocket. From the monotonicity of P it follows that every half-line in the direction of the positive x-axis starting from bd(P) dies not intersect int(P). Therefore the pocket P can be translated to ∞ in the +x direction. But R lies in P'. Therefore, P can also be so translated.

Lemma 3.3: If P is a polygon monotonic in direction θ and R is a simple polygon that does *not* intersect the interior of SH(P, θ + π/2), then P and R are separable with a single translation in direction θ + π/2.

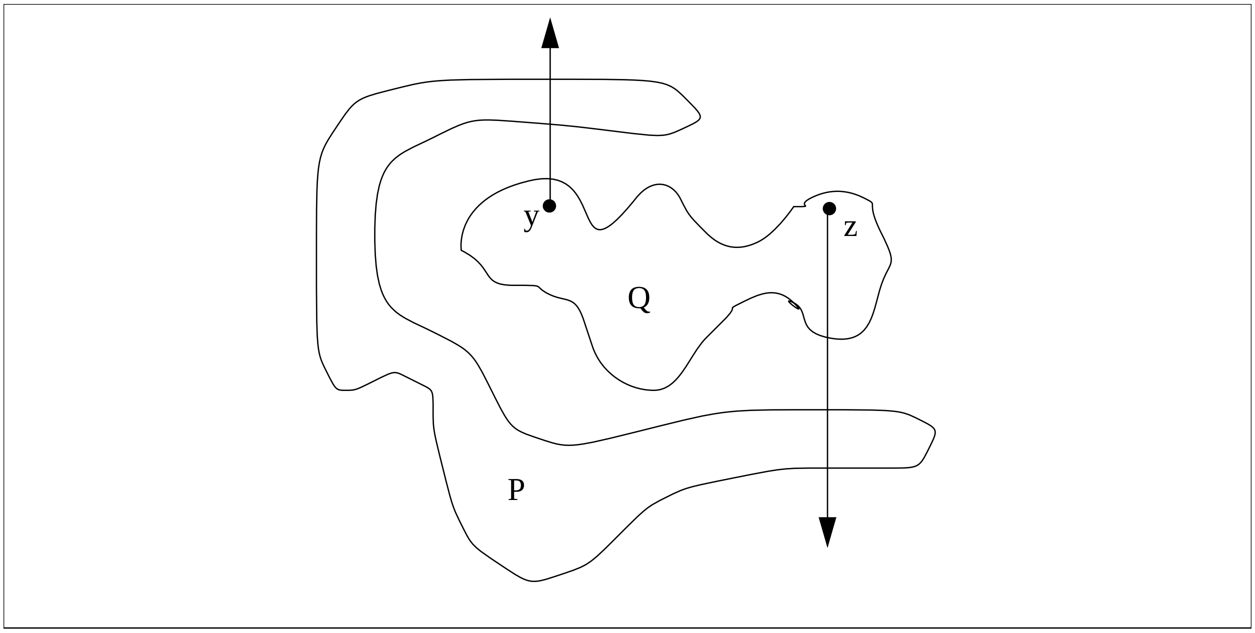

Fig. 7. Another (y) in Q.

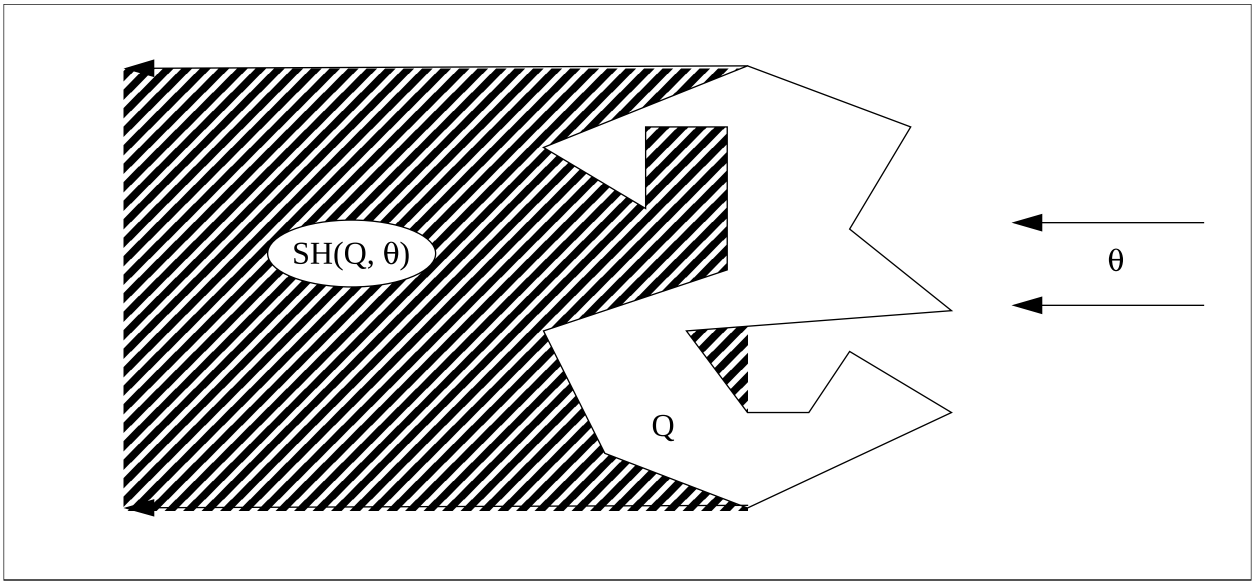

Fig. 5. Illustrating the "shadow" of a polygon.

3. Results

We begin this section by proving some preliminary lemmas.

Lemma 3.1: Two polygons *monotone* in a common direction θ are *movably separable* with a single translation in direction $\theta + \pi / 2$.

Proof: Let P and Q be two polygons monotonic with respect to θ. Without loss of generality assume θ is the x-axis. (Refer to Figure 6.) Let $p_{\min}$ and $p_{\max}$ ($q_{\min}$ and $q_{\max}$) be the vertices of P(Q) with minimum and maximum x coordinates, respectively. If $p_{\max} \le q_{\min}$ or $q_{\max} \le p_{\max}$ then Q can be translated in the ±y directions without colliding with P. Assume that $p_{\max} > q_{\min}$ and $q_{\max} > p_{\min}$ and let $z \in Q$ be a point such that $p_{\max} > x_z > p_{\min}$, where x_z is the x coordinate of z. Clearly z must lie in some triangle of Q determined by three vertices of Q, say $\Delta_{ijk} = (q_i, q_j, q_k)$. Draw the half-lines L_z and U_z in the -y and +y directions, respectively. One of these half-lines must intersect the interior of P. Assume that L_z intersects int(P). We claim that z can move in the +y direction. To see this assume z *cannot* move in the +y direction. This implies that the half-line U_z must intersect int(P). But $z \in \Delta_{ijk} \in Q$ and P and Q do not intersect. Therefore both the lines L_z and U_z intersect int(P), a contradiction since P is *monotonic* in the x direction. Now, if z cannot move downwards we must show that *all* points in Q can move upwards. Let y be any other point in Q. (See Figure 7.) Without loss of generality, assume that x_y is less than x_z. Assume that y *cannot* move upwards. Then U_y must intersect int(P). But L_y intersects int(P), again a contradiction due to the monotonicity of P with respect to the x direction.

Lemma 3.2: If P is a polygon monotonic in direction θ and R is a simple polygon contained in exactly *one* convex-hull pocket of P, then P and R can be separated with a single translation in direction $\theta + \pi/2$.

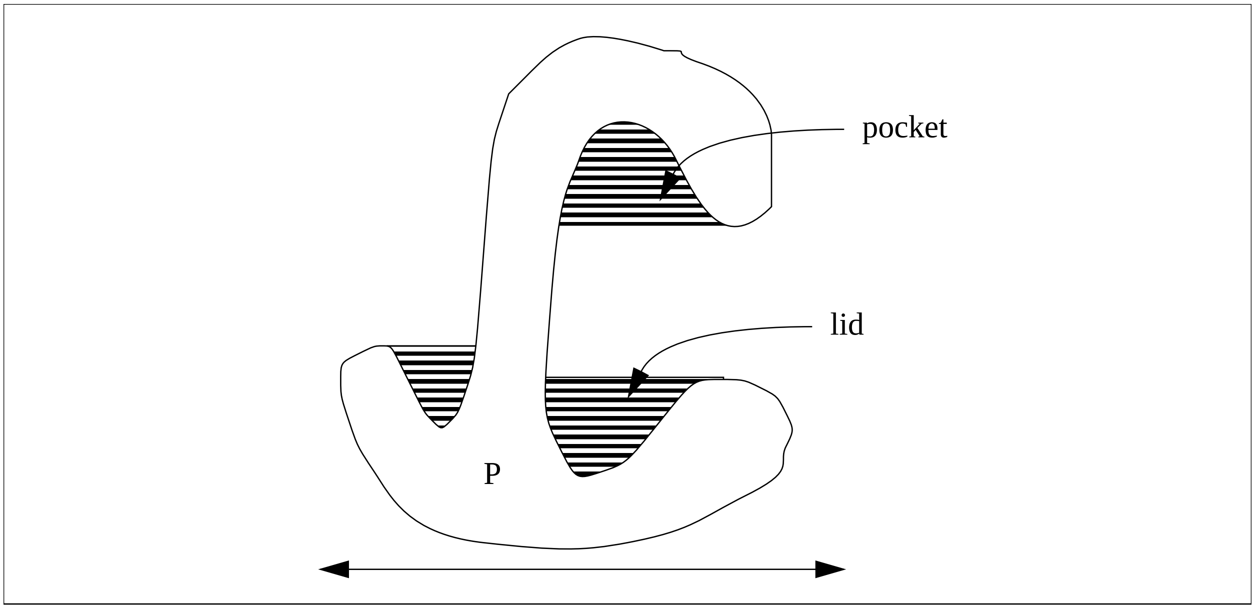

Fig. 3. Illustrating the visibility hull of a polygon.

Definition: Let R be a simple polygon illuminated by a light at ∞ travelling in direction Θ. The *shadow of* Q *in direction* Θ, denoted by SH(Q, Θ), is defined as the unbounded region of the plane not illuminated along with its boundary. (see Figure 5.)

Definition: The *convex hull* of a polygon R, denoted by CH(R), is the minimum-area convex polygon containing R. The *convex deficiency* is the set-difference between CH(R) and R and consists of a set of polygons termed convex-hull pockets.

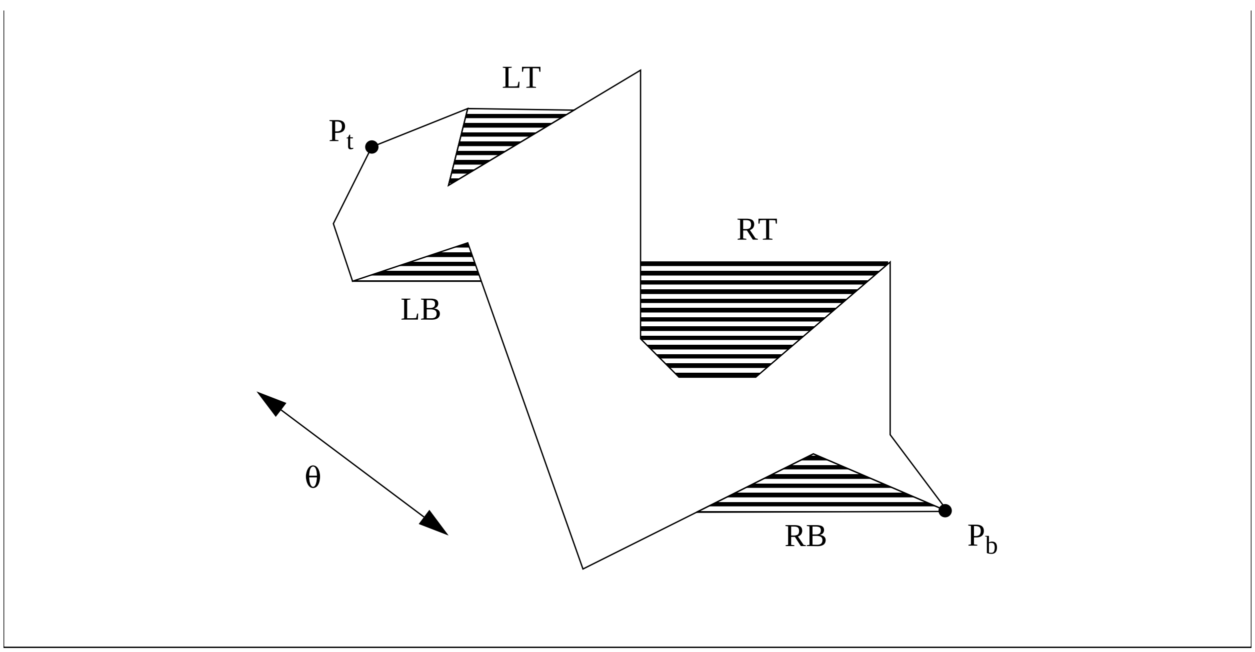

Fig. 4. Four types of pockets.

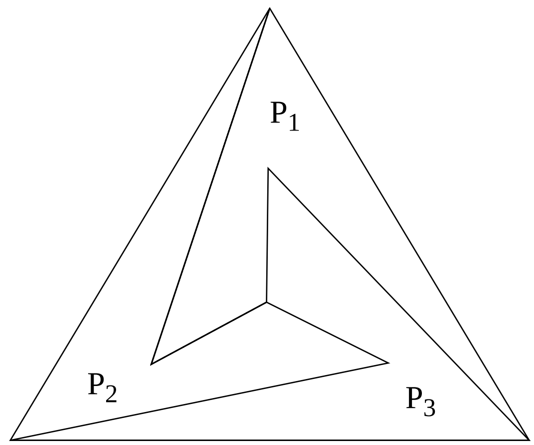

Fig. 2. Three interlocking star-shaped monotone polygons

In this paper we settle the question of the separability of two *monotone* polygons. In section 2 we introduce some notation and definitions. In section 3 we prove that if P and Q are two simple polygons monotonic in directions θ and φ, respectively, then P and Q can be separated with a single translation in at least one of the two directions: $\theta + \pi / 2$, $\varphi + \pi / 2$. Furthermore, such a direction can be determined in linear time. Some concluding remarks are made in section 4.

2. Notation and Definitions

Let $P = \{p_1, p_2,..., p_n\}$ and $Q = \{q_1, q_2,..., q_m\}$ be two simple polygons, i.e. we are given the lists of their vertices, in clockwise order, along with their cartesian coordinates. We assume the polygons are in *standard form,* i.e. the vertices are distinct and no three consecutive vertices are collinear. A pair of vertices, say $p_i p_{i+1}$, defines the i^{th} edge of P. The sequence of vertices and edges forming the boundary of a polygon P, and denoted by bd(P), partitions the plane into two open regions: one bounded, termed the *interior* of P and denoted by int(P), and the other the unbounded *exterior* of P and denoted by ext(P). Let P and Q be monotonic in direction θ and ϕ, respectively, such that int(P) does not intersect int(Q).

Definition: Given a simple polygon R and a direction Θ, the *visibility hull of* R *in direction* Θ, denoted by VH(R, Θ), is the union of R and the closed line segments [a,b] parallel to the direction Θ such that a and b $\in$ R as illustrated in Figure 3. The visibility hull of R in direction Θ can be interpreted as the portions of R visible from $\pm\infty$ in direction Θ. Note that VH(R, Θ) is monotonic with respect to the direction orthogonal to Θ.

Let Θ be the direction of the x-axis. The visibility hull VH(R, Θ) determines four types of pockets: left-top, left-bottom, right-top, and right-bottom. If the lid of a pocket lies above the pocket, then it is a top pocket. Otherwise, it is a bottom pocket. If the lid of a pocket is visible from x = $-\infty$, then the pocket is left. Otherwise, it is a right pocket. These four types of pockets are illustrated in FIgure 4.

tremes among a hierarchy of polygons of varying "complexity" and the pattern recognition literature[20] is filled with problems dealing with classes of polygons which are more complex than *convex* and yet more structured than arbitrary *simple* polygons. Two of the most well known classes of polygons are *star-shaped* and *monotone* polygons. A polygon P is said to be *star-shaped* if there exists a a region K in P, termed the *kernel* of P, such that for all $x \in K$ and all $y \in P$ the line segment joining them lies in P. A polygon P is *monotone* if there exists a direction θ such that the two opposite extreme vertices in direction θ partition the polygon into two polygonal chains each of which, when traversed, yields a monotonically increasing projection onto a line in direction θ.

When considering more than two polygons an interesting question concerning the movable separability of the set is: can one of the polygons always be translated to infinity without disturbing the others? For the case of convex polygons Guibas and Yao[1] showed that the answer is in the affirmative. For star-shaped and monotone polygons Toussaint[3] gave a counterexample with only three polygons as illustrated in Figure 2. Note that the polygons in Figure 2 are both star-shaped and monotonic. The obvious question is: can *two* such polygons interlock? It was proven in ref. 3 that two star-shaped polygons are movably separable with a single translation. More specifically it was shown that if θ is a direction determined by a line which traverses both kernels of the polygons then one polygon can be translated in direction θ while holding the other fixed. This result was independently proven in a more general form by Dawson[8] who showed that the polygons could also be separated by translating them simultaneously with respect to a third fixed point on the plane.

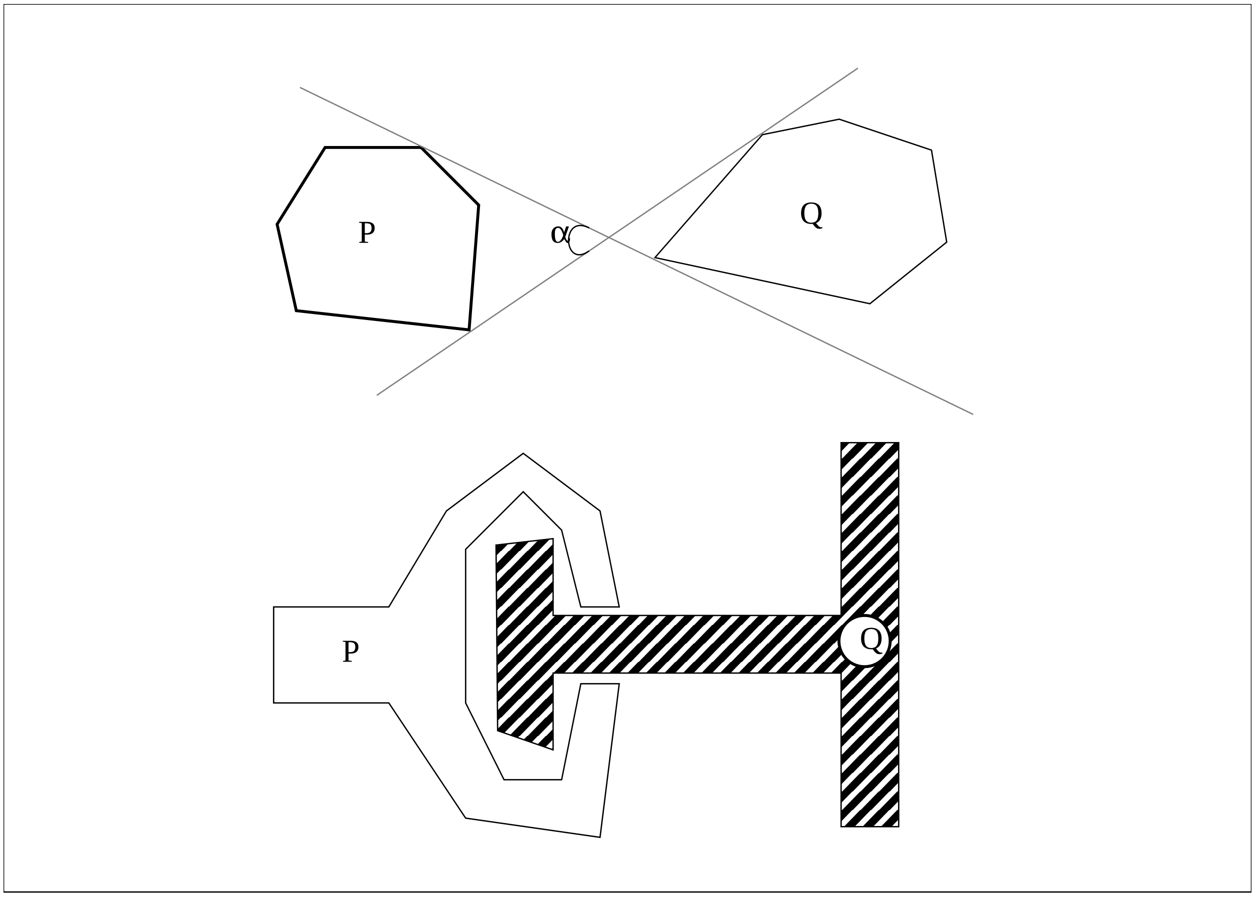

Fig. 1. (a) Two movably separable convex polygons. (b) Two interlocking simple polygons.

Separation of Two Monotone Polygons in Linear Time

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ABSTRACT

Let P = {p_1, p_2,..., p_n} and Q = {q_1, q_2,..., q_m} be two simple polygons monotonic in directions θ and φ, respectively. It is shown that P and Q are separable with a single translation in at least one of the directions: $\theta + \pi / 2$, $\varphi + \pi / 2$. Furthermore, a direction for carrying out such a translation can be determined in $O(m + n)$ time. This procedure is of use in solving the FIND-PATH problem in robotics.

1. Introduction

Spurred by developments is spatial planning in robotics, computer graphics and VLSI layout, considerable attention has been devoted recently to the problem of moving polygons in the plane without collision.[1-11] A typical problem in robotics is the FIND-PATH problem,[12] where a robot must determine if an object, modeled as a polygon in the plane, can be moved from a starting position to a goal state without collisions occurring between the object being moved and the obstacles. Much work has been done on the problem of hypothesizing channels through free space when the obstacles are convex polygons.[13] For nonconvex objects the problem is bypassed by considering the convex hulls of the objects to be the objects themselves. Thus a crucial aspect of robotics for the geometric modeling needed for spatial reasoning and spatial planning is the representation and recognition of the possible types of movements allowed by different non-convex shapes.[14] This paper is a first step in this direction and shows two objects of a certain type of shape, namely monotone polygons, can never interlock. Thus a robot can always separate two such objects without collisions. In fact, this can be done with only a single translation, and the paper presents a simple linear time algorithm for finding a direction for such a motion.

We consider the movable separability of polygons. We say that two polygons P and Q are *movably separable* if one of them can be *moved* to infinity without *colliding* with the other. In this paper the motion being considered is a *single translation*. We say that a collision occurs if at some instant in time during the motion the interiors of P and Q intersect. As an example consider the two polygons P and Q in Figure 1. The two convex polygons in Figure 1(a) are *movably separable* since Q can be translated in any direction excluded by the wedge α. The simple polygons in Figure 1(b) on the other hand are not *movably separable*, i.e. they are *interlocked*. In fact, they are *interlocked* under all displacements, not just translations. One can consider *convex* and *simple* polygons as ex-

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