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Guarding polyhedral terrains

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Abstract

We prove that $\lfloor n/2 \rfloor$ vertex guards are always sufficient and sometimes necessary to guard the surface of an n-vertex polyhedral terrain. We also show that $\lfloor (4n-4)/13 \rfloor$ edge guards are sometimes necessary to guard the surface of an n-vertex polyhedral terrain. The upper bound on the number of edge guards is $\lfloor n/3 \rfloor$ (Everett and Rivera-Campo, 1994). Since both upper bounds are based on the four color theorem, no practical polynomial time algorithm achieving these bounds seems to exist, but we present a linear time algorithm for placing $\lfloor 3n/5 \rfloor$ vertex guards for covering a polyhedral terrain and a linear time algorithm for placing $\lfloor 2n/5 \rfloor$ edge guards.

Keywords: Polyhedral terrains; Art gallery theorems; Matching

1. Introduction

Victor Klee posed the problem of determining the minimum number of guards sufficient to cover the interior of an n-sided art gallery (polygon) in 1973. Chvátal showed that $\lfloor n/3 \rfloor$ guards are sufficient and sometimes necessary to cover the interior of an n-sided art gallery using a lengthy combinatorial argument [4]. Subsequently Fisk [9] gave a concise and elegant proof using the fact that the vertices of a triangulated polygon may be three-colored. Avis and Toussaint [2] used Fisk's proof to design an $O(n \log n)$ algorithm for placing the guards. Recently, Kooshesh and Moret [12] showed that the guards can be placed in linear time. Although many similar problems have been studied in computational

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geometry [14–16], little is known about guarding an object in three dimensions. In this paper, we present some results on guarding the surface of a polyhedral terrain.

The problem of guarding a polyhedral terrain was first investigated by deFloriani et al. [7]. They showed that finding the minimum number of guards could be done using a set covering algorithm. Cole and Sharir [6] showed that the problem was NP-complete. Goodchild and Lee [10] and Lee [13] present some heuristics for placing vertex guards on a terrain.

In this paper, we show that $\lfloor n/2 \rfloor$ vertex guards are always sufficient and sometimes necessary to guard an n-vertex terrain. We also present a linear time algorithm for placing $\lfloor 3n/5 \rfloor$ vertex guards to cover a terrain. With respect to edge guards, we establish that $\lfloor (4n-4)/13 \rfloor$ edge guards are sometimes necessary to guard the surface of an n-vertex terrain. The sufficiency result of $\lfloor n/3 \rfloor$ edge guards is proved by Everett and Rivera-Campo [8]. We show that there is no gap between the upper and lower bounds for edge guards when considering planar graphs. Finally, we present a linear time algorithm for placing $\lfloor 2n/5 \rfloor$ edge guards to cover a polyhedral terrain. Reducing the gap between sufficiency and necessity for edge guards and finding efficient, practical algorithms to achieve the known bounds remain open problems.

2. Visibility on polyhedral terrains

We begin by reviewing some of the terminology used throughout this paper.

We define a terrain T as a triangulated polyhedral surface with n vertices $V = \{v_1, v_2, \ldots, v_n\}$. Each vertex v_i is specified by three real numbers (x_i, y_i, z_i) which are its cartesian coordinates and z_i is referred to as the height of vertex v_i. It is convenient to assume that z_i is nonnegative so that if the X–Y plane is associated with *sea-level*, no points on the terrain are below sea-level. Let $P = \{p_1, p_2, \ldots, p_n\}$ denote the orthogonal projections of the points $V = \{v_1, v_2, \ldots, v_n\}$ on the X–Y plane, i.e., each point p_i is specified by the two real numbers (x_i, y_i). It is assumed that the set $P = \{p_1, p_2, \ldots, p_n\}$ is in general position, i.e., no three points are collinear and no four are cocircular so that the projections of the edges of the polyhedral surface onto the X–Y plane determine a triangulation of P (hence the term triangulated polyhedral surface). We refer to the triangulation as the underlying triangulated planar graph associated with the terrain. Therefore we can view a terrain T as the graph of a polyhedral function $z = F(x, y)$, defined over $\mathrm{CH}(P)$. Sometimes a polyhedral terrain is assumed to be a monotone polyhedral surface, i.e., a polyhedral surface having exactly one intersection with every vertical line [6]. In our case we assume the stronger condition that the intersection of every vertical line in the interior of $\mathrm{CH}(P)$ with the polyhedral surface is a single point. Intuitively, a monotone terrain admits vertical walls whereas our definition does not. Since the orthogonal projection of T onto the X–Y plane is a planar straight-line subdivision or map, it follows that T has $O(n)$ edges and $O(n)$ triangular faces.

Two points a, b on or above T are said to be *visible* if the line segment $\overline{ab}$ does not intersect any point strictly below T. Given a point (guard) p on or above T, the subset of points of T that are visible from p is called the visible region of T from p and is denoted by $\mathrm{VR}(T|p)$.

Throughout this paper, we only consider problems concerning vertex and edge guards. A *vertex guard* is a guard that is only allowed to be placed at the vertices of T. An *edge guard* is a guard that is only allowed to be placed on the edges of T. A point x on T is said to be visible to an edge if there exists a point y on the edge such that x and y are visible.

A set of guards covers a terrain if every point on the terrain is visible from at least one guard in the set. The vertex guarding problem we study is the following: what is the number of vertex guards that are always sufficient and sometimes necessary to cover any polyhedral terrain? Similarly, the edge guarding problem is to determine the number of edge guards that are always sufficient and sometimes necessary to cover any polyhedral terrain.

The combinatorial counterparts of these terrain guarding problems can be expressed as guarding problems on the planar triangulated graph underlying the terrain. A vertex guard on the graph can only guard the faces adjacent to that vertex, and an edge guard on the graph can only guard the faces adjacent to the endpoints of the edge. It seems difficult to show that the problem of guarding a polyhedral terrain is equivalent to the combinatorial problem of guarding the underlying planar triangulated graph. However, a valid placement of vertex (respectively edge) guards on the underlying planar graph is also a valid placement of vertex (respectively edge) guards on the polyhedral terrain since a guard on the terrain can always see the faces adjacent to it. Therefore, an upper bound on the number of guards used to guard a triangulated planar graph is also an upper bound for polyhedral terrains. The difficulty comes in proving lower bounds, since a vertex guard on a polyhedral terrain may see more than just the faces adjacent to that vertex. We circumvent the difficulty by providing lower bound constructions on *convex* terrains, which by convexity have the property that a vertex can only see the faces adjacent to it.

Definition 2.1. A polyhedral terrain T is a *convex* terrain provided that T is a terrain and every point on T is also a point on the boundary of the convex hull of the vertices of T.

The approach used to prove the lower bounds for the terrain guarding problems is to first construct a triangulated planar graph that achieves the desired bound for the combinatorial guarding problem, and then show that the given construction can be realized as a convex terrain. The main building blocks used to show that the lower bound constructions can be realized as a convex terrain are *truncation* and *stellation*. *Truncation* refers to the removal of a vertex from a convex polyhedron while retaining convexity and *stellation* refers to the addition of a vertex to a convex polyhedron while retaining convexity. The following two theorems from polyhedral theory provide the necessary tools. Although the following theorems hold for d-polytopes in $\mathbb{R}^d$, we restrict our attention to the case where $d = 3$. Before stating the theorems, we review some terminology. For more details, the reader is referred to [3] or [11].

Given a convex polyhedron P, we denote its open interior by $\operatorname{int}(P)$, its open exterior by $\operatorname{ext}(P)$ and its boundary by ∂P. The boundary is considered part of the polyhedron, i.e., $P = \partial P \cup \operatorname{int}(P)$. A face F of a polyhedron P is a vertex, an edge or a facet of P. The dimension of a face F, denoted $\dim F$, is 0 if F is a vertex, 1 if F is an edge and 2 if F is a facet. The linear subspace containing F is denoted $\operatorname{aff} F$. Given a convex polyhedron P, a point v, and a plane H such that $H \cap \operatorname{int}(P) = \emptyset$, we say that v is *beneath* H, or *beyond* H (with respect to P), provided v belongs to the open halfspace determined by H which contains $\operatorname{int}(P)$ or does not meet P. Given a set of points S in $\mathbb{R}^3$, $\operatorname{CH}(S)$ denotes the convex hull of the points.

Theorem 2.2 [3, Theorem 11.11]. *Let P be a convex polyhedron in $\mathbb{R}^3$. Let V represent the vertices of P. Let H be a plane in $\mathbb{R}^3$ with $H \cap \operatorname{int}(P) \neq \emptyset$, and $H \cap V = \emptyset$, and let K be one of the two closed halfspaces bounded by H. Then we have:*

1. *The set $P' = K \cap P$ is a convex polyhedron and $H \cap P$ is a facet of P'.*
2. *If F is a face of P such that $K \cap F \neq \emptyset$, then $F' = K \cap F$ is a face of P', and* $\dim F' = \dim F$.
3. *Let F' be a face of P'. Then either F' is a face of the facet $H \cap P$, or there is a unique face F of P such that $F' = K \cap F$.*

From the above theorem, we conclude that given a convex terrain T, a vertex v of T can always be truncated such that the resulting object T' is a convex terrain which is the same as T except for the modification of the faces adjacent to v and the new facet created by the truncation of v.

Theorem 2.3 [11, Theorem 5.2.1]. *Let P and P' be two convex polyhedra in $\mathbb{R}^3$, and let V be a vertex of P', and $V \notin P$, such that $P' = \mathrm{CH}(V \cup P)$. Then*
1. *A face F of P is a face of P' if and only if there exits a facet F' of P such that $F \subset F'$ and V is beneath F'.*
2. *If F is a face of P then $F' = \mathrm{CH}(V \cup F)$ is a face of P' if and only if either $V \in \mathrm{aff} F$, or among the facets of P containing F there is at least one such that V is beneath it and at least one such that V is beyond it.*

Moreover, each face of P' is of one and only one of those types.

From the above theorem, we conclude that given a convex terrain T, a vertex v can always be added to T such that the resulting object T' is a convex terrain, which is the same as T except for vertex v and the faces adjacent to v.

3. Guards on a terrain

In this section, we show that $\lfloor n/2 \rfloor$ vertex guards are always sufficient and sometimes necessary to guard a polyhedral terrain. We also show that $\lfloor (4n-4)/13 \rfloor$ edge guards are sometimes necessary to guard a polyhedral terrain.

3.1. Vertex guards

Lemma 3.1. *The seven-vertex graph shown in Fig. 1 needs at least three vertex guards. Furthermore, if three vertex guards are used to cover it, then at most one of the three guards can be an exterior vertex.*

Proof. Suppose that two vertices suffice. One of the inner four vertices must be chosen to cover the inner triangles. If the central vertex is chosen, then the remaining unguarded (outer layer) triangles cannot be covered by one guard, as the triangles A and B do not share a vertex. Therefore, one of the three middle vertices must be chosen. Without loss of generality, suppose vertex x is chosen. Then, the unguarded triangles (A and the three triangles adjacent to A) are not coverable by one vertex guard.

Now we show that at most one vertex guard can be an exterior vertex. If all three were exterior vertices, then the middle three triangles would be unguarded. Suppose that at least two of the vertex guards are exterior vertices. Without loss of generality, let them be the bottom two. We now have A and the three central triangles (directly below A) unguarded. These triangles cannot be guarded with one additional guard. □

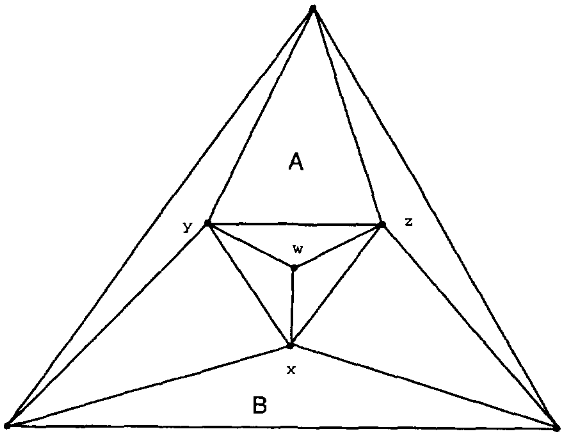

Fig. 1. A seven-vertex graph.

Using the graph in Fig. 1, we construct a series of planar subdivisions $S_1, \ldots, S_k$, where S_1 is the graph of Fig. 1 and S_{k+1} is obtained from S_k in the following manner: let S_{k+1} be the graph of Fig. 1 with one of the central triangles replaced by a copy of S_k (without loss of generality, suppose it is the one below face A). We show the following property about S_k.

Lemma 3.2. *S_k is triangulated, has $n_k = 4k + 3$ vertices, needs $g_k = 2k + 1$ guards, and if it is covered by exactly $2k + 1$ guards, then at most one guard is on the exterior face.*

Proof. By induction on k.
Basis: $k = 1$. Follows from Lemma 3.1.
Inductive Hypothesis: For all $k \leqslant t$, $t \geqslant 1$, S_k is triangulated, has $n_k = 4k + 3$ vertices, needs $g_k = 2k + 1$ guards, and if it is covered by exactly $2k + 1$ guards, then at most one guard is on the exterior face.
Inductive Step: $k = t + 1$. S_{t+1} is triangulated by construction. It has $n_t + 4 = (4t + 3) + 4 = 4(t + 1) + 3$ vertices. We now only need to show that it requires $2(t + 1) + 1 = 2t + 3$ guards, and that if it uses exactly $2t + 3$ guards, then only one exterior vertex is a guard.
In S_{t+1}, there is a copy of S_t. By induction, this copy of S_t must use at least $2t + 1$ guards. We consider cases based on how many guards this copy of S_t uses.
Case 1: The copy of S_t uses exactly $2t + 1$ guards. Then the copy of S_t has at most one guard on one of its exterior vertices. There are 4 cases: no guard is placed on the exterior of S_t, left vertex (y) is a guard, right vertex (z) is a guard, and the lower vertex (w) is a guard.

Case 1.1: No guard is placed on the exterior of S_t. Since S_t is already covered, two guards suffice to cover the remainder of S_{t+1}. We have that $g_{t+1} = (2t + 1) + 2 = 2(t + 1) + 1$. If exactly 2 guards are used, then at most one of them can be on the exterior of S_{t+1}.

Case 1.2: A guard is placed at y. This configuration requires at least 2 guards. If covered with exactly two guards ($(2t + 1) + 2 = 2t + 3$ guards total), then at most one is on the exterior face.

Case 1.3: A guard is placed at z. This case is symmetric to Case 1.2.

Case 1.4: A guard is placed at w. There is a ring of six triangles that requires two guards and at most one of these guards is on the exterior face.

Case 2: The copy of S_t uses exactly $2t+2$ guards. Then the copy of S_t may have guards on all three of its exterior vertices (i.e., w, y, z). However, this still leaves one face (B) uncovered, so one more guard is required. If only one more guard ($2t + 3$ total) is used, then only that guard may be on the exterior face.

Case 3: The copy of S_t uses more than $2t + 2$ guards. Then the induction hypothesis is true. □

As seen above, the seven-vertex graph forms the basis of the lower bound construction. To provide a lower bound for terrains, we show how to construct a convex terrain whose underlying graph is S_k. We begin by showing that a convex terrain whose underlying graph is the seven-vertex graph of Fig. 1 can be constructed. This construction will be referred to as *Construction A*.

Consider a regular tetrahedron, with one horizontal facet F and apex a (see Fig. 2). The initial convex terrain is the surface of this tetrahedron except for the facet F. The first step is to truncate the apex (vertex a) with a horizontal plane. This results in a triangular facet with vertices b, c and d as shown. The next three steps involve truncating these three vertices. Vertex b is truncated with a plane defined by vertex 1 and the mid-points of edges $\overline{bc}$ and $\overline{bd}$. The other two vertices are truncated similarly. Finally, vertex h is stellated on facet *efg*. This process completes the construction of a convex terrain whose underlying graph is S_1. To construct one whose underlying graph is S_2, simply construct S_1 on facet *efh* and so on.

Theorem 3.3. *There exists a terrain on n vertices, for any $n \equiv 3 \bmod 4$ that requires $\lfloor n/2 \rfloor$ vertex guards.*

Proof. Follows directly from Lemma 3.2 and Construction A. For that terrain, we have:

$$g_k = 2k + 1 \quad \text{and} \quad n_k = 4k + 3, \quad \text{therefore}$$

$$g_k = 2\big((n_k - 3)/4\big) + 1 = \big((n_k - 3)/2\big) + 1 = (n_k - 1)/2 = \lfloor n_k/2 \rfloor. \qquad \square$$

Theorem 3.4. *$\lfloor n/2 \rfloor$ vertex guards are always sufficient and sometimes necessary to guard the surface of an arbitrary terrain T with n vertices.*

Proof. First 4-color the vertices of T'. This can always be done since T' is a planar graph [1]. By the pigeon hole principle, among the 4 colors there must be 2 colors such that no more than $\lfloor n/2 \rfloor$ vertices are colored by these two colors. Furthermore, these $\lfloor n/2 \rfloor$ vertices are sufficient to guard all of the faces of T' (because every triangle must have at least one vertex colored with one of these 2 colors). Necessity follows from Theorem 3.3. □

3.2. Edge guards

We now consider the problem of guarding a polyhedral terrain using edge guards.

Lemma 3.5. *The graph in Fig. 3 needs at least two edge guards. Furthermore, if a mixture of edge guards and vertex guards are allowed, then one edge guard and one vertex guard suffice.*

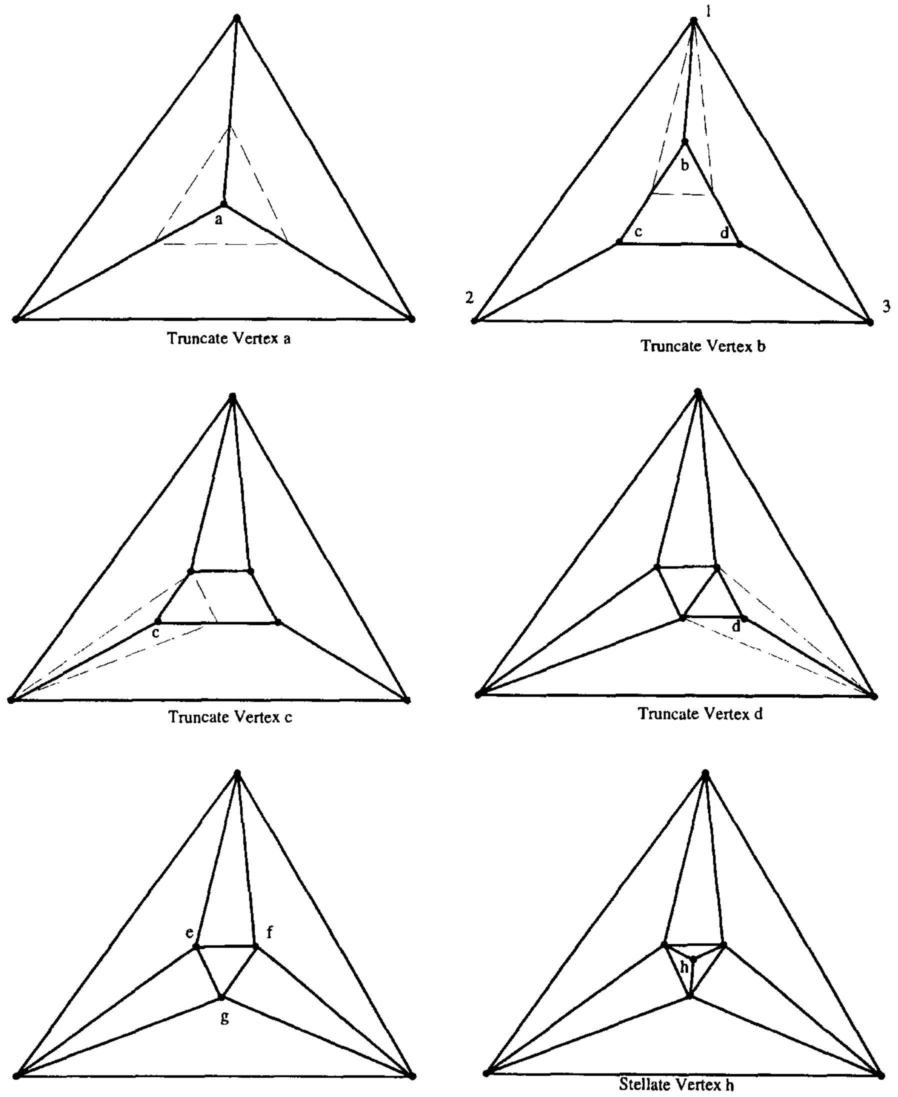

Fig. 2. Building a convex seven-vertex terrain.

Proof. Suppose one edge guard suffices. We then have the following cases.

(i) $\overline{ab}$, $\overline{ac}$, $\overline{bc}$ do not cover triangle(x, y, z).

(ii) $\overline{ay}$, $\overline{ax}$ do not cover triangle(b, c, z).

(iii) $\overline{xz}$ does not cover triangle(a, y, c).

All other cases follow by symmetry. Therefore we need at least 2 edge guards for the graph in Fig. 3 (edges $\overline{ab}$ and $\overline{yz}$ suffice). For all the cases above the unguarded faces can be covered by one vertex guard. □

Theorem 3.6. *There is a planar triangulation that needs at least* $(4n - 4)/13$ *edge guards.*

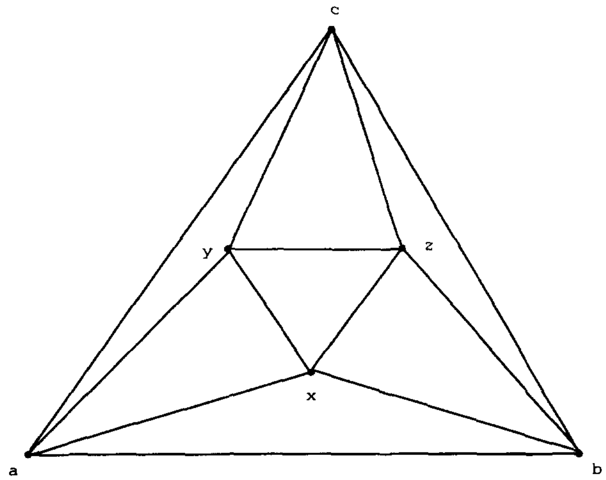

Fig. 3. A 6-vertex graph which needs 2 edge guards.

Proof. Such a planar triangulation is derived from an arbitrary triangulated convex polygon P with v vertices and $v-2$ internal triangular faces.

We put a copy of Fig. 3 in each face of P and along each edge of the boundary of P. Then we triangulate the untriangulated faces. (In total we add $v+(v-2)=2v-2$ such copies to P.) Suppose the triangulation we get is P^* and it needs g_e edge guards. Because guards cannot be shared between any copies of Fig. 3, P^* requires at least $g_e = 2(2v-2) = 4v-4$ edge guards and has $v_{P^*} = v + 6(2v-2) = 13v - 12$ vertices. Substituting v_{P^*} by n, we have

$$g_e = (4n-4)/13. \qquad \square$$

This construction on graphs translates into a lower bound for terrains through the following construction of a convex terrain whose underlying graph is the planar triangulated graph in the proof of Theorem 3.6. Refer to Fig. 4 for the construction to follow.

We begin with a set of n points in the X–Y plane placed in the following manner. One point labelled 0 is placed at the origin. Let C be a unit radius circle on the X–Y plane centered at the origin. The other $n-1$ points (labelled $2, 4, 6, \ldots, 2(n-1)$) are placed in clockwise order on the boundary of C, but are contained in a quarter circle. Now raise point 0 slightly (i.e., set its Z-coordinate to some positive value, say $1/2$), and compute the convex hull of the set of points. Our initial terrain T_0 consists of the facets of the convex hull that have an outer normal with positive Z-component. This is the first phase of the construction which achieves a terrain whose underlying graph is a triangulated convex polygon.

We continue the construction by expanding the terrain in the following way. For every consecutive pair of points on the boundary of the quarter circle, place a point on the boundary of the circle half-way between the two points. Label the points $3, 5, 7, \ldots, 2(n-1)-1$. Place point 1 slightly to the left of the midpoint of edge $\overline{0,2}$ (i.e., at coordinate $(-\varepsilon, 1/2, 0)$ for some small positive ε) and point $2n-1$ slightly below edge $\overline{0,10}$ (i.e., at coordinate $(1/2, -\varepsilon, 0)$). Now set the Z-coordinate of all the newly added points to $-1/2$. Compute the convex hull. The resulting terrain T_1 is the set of facets of the convex hull that have an outer normal with positive Z-component.

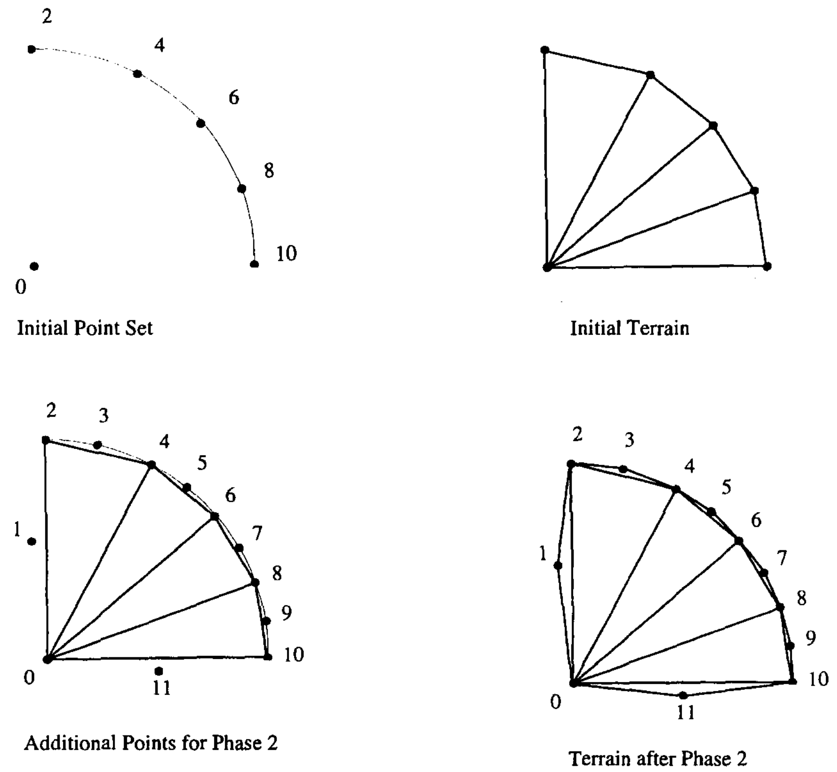

Fig. 4. Illustration for construction.

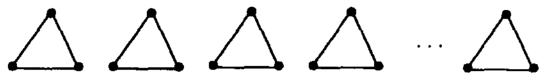

Fig. 5. A planar graph that needs $n/3$ edge guards.

To complete the construction place a copy of Fig. 3 on each facet of the terrain T_1 by using the first five steps of Construction A for each facet. We conclude with the following.

Corollary 3.7. *There exists a terrain on n vertices, for any $n \equiv 1 \bmod 13$ that requires $\lfloor (4n-4)/13 \rfloor$ edge guards.*

Everett and Rivera-Campo [8] have shown that $\lfloor n/3 \rfloor$ edge guards are sufficient to cover a polyhedral terrain. In fact, they prove the result by showing that $\lfloor n/3 \rfloor$ guards are sufficient to cover a planar triangulated graph. Thus there is a gap between the upper and lower bounds for edge guards on a polyhedral terrain. But, if we simply look at planar graphs, then the gap no longer exists.

Consider a planar graph consisting of disjoint triangles (see Fig. 5). Such a planar graph requires at least one edge guard per triangle.

We can modify the construction in Fig. 5 to give a planar graph that is almost triangulated needing $n/3$ edge guards. The graph in Fig. 6 is a two-connected planar graph where every face is either a

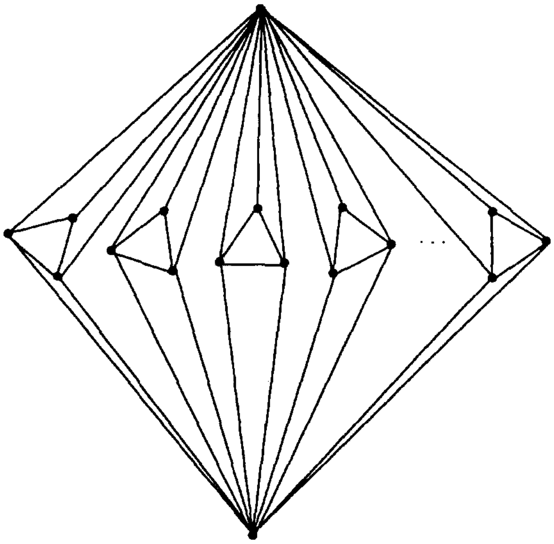

Fig. 6. A two-connected planar graph that needs $n/3$ edge guards.

triangle or a quadrilateral. Since there is a set of $(n-2)/3$ disjoint triangles each needing one edge guard, we have the following theorem.

Theorem 3.8. *There exists a two-connected planar graph on n vertices, for any $n \equiv 2 \bmod 3$ that requires $\lfloor n/3 \rfloor$ edge guards.*

4. Algorithms for placing terrain guards

In this section, we present some practical, efficient algorithms for placing the vertex and edge guards. Since establishing the number of vertex guards and the number of edge guards sufficient to cover a terrain required the use of the four color theorem, finding a practical efficient algorithm to place the guards seems unlikely unless a deeper understanding of the problem is achieved. To this end, we present practical algorithms for guard placement which approximate the upper bounds.

4.1. Placing vertex guards

Observation 4.1. *Given a five coloring of the vertices of any terrain, any set of three color classes provides a vertex guarding of the terrain since every face of the terrain is a triangle except possibly the outer face (i.e., the outer face of the underlying planar graph which need not be guarded).*

Based on this observation, a simple linear time algorithm follows.

Algorithm 1.
1. *5-color the vertices of the planar triangulation graph;*
2. *Among the 5 colors, choose 3 colors which are minimally used.*

By the result of [5], step 1 is $O(n)$. $O(n)$ time also suffices for step 2. Therefore, the complexity of Algorithm 1 is $O(n)$.

4.2. Edge guard placement

We extend some of the elegant ideas of Everett and Rivera-Campo [8] in order to develop a linear time algorithm for placing $\lfloor 2n/5 \rfloor$ edge guards to cover a polyhedral terrain. We use the following lemma.

Lemma 4.2. *Given a finite collection of R real numbers there exists an element of R that must be less than or equal to the average.*

Proof. Let k be the average of the collection R. Suppose that there were no elements of R that were less than or equal to k. This implies that all of the elements are greater than k. But then k could not be the average. □

Our edge guard algorithm proceeds as follows. The first step in the algorithm is to five color the vertices. Let the five colors be: 1, 2, 3, 4, 5.

Let $\mathrm{Matching}(a, b, c)$ denote a *maximal* matching (which is not necessarily a *maximum* matching) on the graph induced by the vertices in the three color classes a, b and c. Although $\mathrm{Matching}(a, b, c)$ does not provide a set of edges that guards the whole terrain, if we take all the edges in $\mathrm{Matching}(a, b, c)$ as well as one edge from each of the remaining unmatched vertices of color a, b and c then we guard the whole terrain by Observation 4.1. Let $\mathrm{Guard}(a, b, c)$ represent the size of a set of edge guards obtained in this way. Also, let $\mathrm{Size}(a, b, c)$ represent the number of vertices of the three color classes a, b and c. We have the following relation: $\mathrm{Guard}(a, b, c) = \mathrm{Size}(a, b, c) - \mathrm{Matching}(a, b, c)$. This relation holds because for each edge of the matching, we reduce the number of unmatched vertices by 2 which results in a reduction of the size of guard by 1.

There are 10 possible combinations of three color classes resulting from the five coloring of the graph. We list them here in lexicographic order for reference: 123, 124, 125, 134, 135, 145, 234, 235, 245, 345. Let c_i represent the ith combination in lexicographic order. Notice that each color class appears in 6 combinations. Thus,

$$\sum_{i=1}^{10} \mathrm{Size}(c_i) = 6n.$$

Therefore we have the following lemma.

Lemma 4.3. *If $\sum_{i=1}^{10} \mathrm{Matching}(c_i) \geqslant 2n$, then there exists a guarding of size $\leqslant \lfloor 2n/5 \rfloor$.*

Proof. The average size of guard

$$= \frac{1}{10}\left(\sum_{i=1}^{10} \mathrm{Size}(c_i) - \sum_{i=1}^{10} \mathrm{Matching}(c_i)\right) \leqslant \frac{6n - 2n}{10} = \frac{2n}{5}.$$

Therefore, one of the combinations provides a guarding of size $\leqslant 2n/5$ by Lemma 4.2. □

When $\sum_{i=1}^{10}$ Matching$(c_i) \leqslant 2n$, we have the following lemma.

Lemma 4.4. *One of the following pairs of* matchings *provides a set of edges that guards the whole terrain:* Matching$(1,2,3)$ *and* Matching$(1,4,5)$, Matching$(1,2,5)$ *and* Matching$(2,3,4)$, Matching$(1,2,4)$ *and* Matching$(3,4,5)$, Matching$(1,3,4)$ *and* Matching$(2,3,5)$, Matching$(1,3,5)$ *and* Matching$(2,4,5)$.

Proof. Let us first consider the first pair of matchings. Suppose there is a triangle which is not guarded. This means that all three vertices of the triangle must be unmatched. Clearly, the triangle cannot contain edges whose endpoints have colors: $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{1,5\}$, $\{2,3\}$, $\{4,5\}$, because if it did, we could add an extra edge to one of the matchings contradicting the fact that it is maximal. So it must contain one of: $\{2,4\}$, $\{2,5\}$, $\{3,4\}$, $\{3,5\}$. Suppose it contained $\{2,4\}$. Well the third vertex must have color: 1, 3 or 5. Thus, the triangle contains an edge which must be guarded. If it did not we could add an extra edge to one of the two matchings contradicting the fact that they are maximal. The argument is similar for the other three $\{2,5\}$, $\{3,4\}$, $\{3,5\}$. The argument for the other four matching pairs is also similar. □

The average size of a matching pair

$$= \sum_{i=1}^{10} \frac{\text{Matching}(c_i)}{5} \leqslant \frac{2n}{5}$$

(note that the average is taken over five since there are five matching pairs). Thus, one of the pairs of matchings provides a guarding with the desired size by Lemma 4.2.

Computing a maximal matching on a graph induced by the three chosen colors can be done in linear time in the number of edges in the graph. Thus $O(n)$ time suffices to compute all of the matchings induced by all 10 combinations of three color classes. Once all of the matchings are computed, Lemmas 4.3 and 4.4 guarantee that either a guarding or a pair of matchings will have size less than or equal to $2n/5$. Since there are only 10 different guardings and 5 pairs of matchings, the appropriate set can be found in only linear time. Therefore, we have the following theorem.

Theorem 4.5. *Given a polyhedral terrain on n vertices, $O(n)$ time is sufficient to find a set S of edges to guard the terrain, where $\|S\| \leqslant 2n/5$.*

5. Closing remarks and open problems

The following table summarizes the results of guarding polyhedral terrains.

	Sufficiency	Necessity	Algorithmic bounds
Vertex guards	$\lfloor n/2 \rfloor$	$\lfloor n/2 \rfloor$	$\lfloor 3n/5 \rfloor$
Edge guards	$\lfloor n/3 \rfloor$	$\lfloor (4n-4)/13 \rfloor$	$\lfloor 2n/5 \rfloor$

There are three open problems related to these problems:

(1) Is it possible to reduce the gap between sufficiency and necessity for edge guards?

(2) Are there practical efficient algorithms that match the known bounds?

Note. We have tried to use a computer search to find whether there is a 9-vertex planar triangulation that needs 3 edge guards. If one exists, we can immediately improve the lower bound on edge guards to $(6n - 6)/19$. We used the Mathematica package by Komei Fukuda to generate all triangulations for a 9-vertex random point set and checked whether 2 edge guards sufficed. We ran the program on several hundred random point sets but no such triangulation was found up to this writing. Thus, we raise the following problem to conclude our paper:
(3) Is there a 9-vertex planar triangulation which needs 3 edge guards?

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