Simple Induction Proof of the Arithmetic Mean – Geometric Mean Inequality

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The Arithmetic Mean – Geometric Mean Inequality: Induction Proof

The Arithmetic-Geometric mean inequality: if $a_1, a_2, \ldots, a_n > 0$,

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}$$

where the equality holds if, and only if, all the a_i's are equal.

Base Case: For $n = 2$ the problem is equivalent to

$(a_1 + a_2)^2 \geq 4a_1a_2$, which is equivalent to $(a_1 - a_2)^2 \geq 0$.

Or alternately expand: $\left(\sqrt{a_1} - \sqrt{a_2}\right)^2$

Kong-Ming Chong, "The Arithmetic Mean-Geometric Mean Inequality: A New Proof," *Mathematics Magazine*, Vol. 49, No. 2 (Mar., 1976), pp. 87-88.

The Arithmetic Mean – Geometric Mean Inequality: Induction Proof – *continued…*

Induction Hypothesis: Assume the statement is true for *n*-1.

Proof: Without lost of generality assume that

$$a_1 \le a_2 \le \cdots \le a_n.$$

Let G be the geometric mean $G := \sqrt[n]{a_1 a_2 \cdots a_n}$. Then it follows that

$a_1 \le G \le a_n$. Note that since

$$a_1 + a_n \ge \frac{a_1 a_n}{G} + G$$

$$a_1 + a_n - G - \frac{a_1 a_n}{G} = \frac{a_1}{G}(G - a_n) + (a_n - G) = \frac{1}{G}(G - a_1)(a_n - G) \ge 0$$

The AG-Mean Inequality Induction Proof – *continued*...

By the induction hypothesis

$$\frac{a_2 + \cdots + a_{n-1} + \frac{a_1 a_n}{G}}{n-1} \geq \sqrt[n-1]{G^n/G} = G$$

Hence

$$a_2 + \cdots + a_{n-1} + \frac{a_1 a_n}{G} \geq (n-1)G$$

and

$$\frac{a_2 + \cdots + a_{n-1} + \frac{a_1 a_n}{G} + G}{n} \geq G$$

But since

$$a_1 + a_n \geq \frac{a_1 a_n}{G} + G$$

it follows that

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq G$$