Unraveling Roman mosaic meander patterns: A simple algorithm for their generation

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Abstract:

A geometrical analysis of the meander decorative patterns on a Roman pavement mosaic found at the Roman villa in Chedworth, England, is presented. The analysis reveals that the intricate swastika meander pattern consisting of four closed curves could have been easily constructed using a very simple hypothesized algorithm. The algorithm also explains the design of Roman swastika meanders found throughout the Roman Empire. Connections are indicated between these patterns and the *sona* traditional art of Angola as well as the *kolam* traditional art of Tamil South India. The analysis and algorithm described have applications to the classification of geometric mosaic patterns, the design of new patterns, and the reconstruction of mosaics that have been partially destroyed by the ravages of time.

1. Introduction

One of the most impressive Roman villas in the United Kingdom is the Chedworth Villa in Gloucestershire. A detailed description of the villa, the history of its discovery, and useful references have been compiled by P. Bethel and published by the The National Trust [3]. Several uncovered geometric mosaics there have been partially restored since its discovery. The most fascinating mosaic, found in the dining room, contains an intricate swastika meander pattern, pictured in Figure 1.

Figure 1: The geometric mosaic at Chedworth.

Swastika meander patterns such as the one pictured in Figure 1 are common in ancient Roman and Greek mosaic patterns (as well as Chinese decorations [31]), and provide design techniques for both "two-dimensional" decoration of areas, and "one-dimensional" frieze or fret patterns [2], [5-8], [13-15], [19], [23], [29]. However, the scholarly study of Roman and Greek mosaics conspicuously overlooks a geometrical analysis of these geometric patterns, and tends to focus primarily on those aspects of the mosaics that deal with plants or human figures, in order to draw conclusions about the traditions and aspirations of the societies that created these designs [7], [15], [21]. Notable exceptions are the work of Phillips on the topology of Roman mosaic mazes [24], as well as the geometric analyses of Smith [26] and Sutton [27]. In this paper we break with this tradition and provide a geometric analysis of one of the most fascinating geometric mosaics in Roman Britain. The analysis reveals that the intricate swastika meander pattern found on the Chedworth mosaic, consisting of four closed curves, could have been easily constructed using a very simple hypothesized algorithm. The algorithm also explains the design of Roman and Greek swastika meanders found throughout the Roman Empire. Connections are indicated between these meanders and the *sona* traditional art of Angola as well as the *kolam* traditional art of Tamil South India. Previously Gerdes [10] provided a comprehensive comparative analysis of sona drawings with designs from Ancient Egypt, Mesopotamia, the Vanuatu Islands and Celtic knots. We may now add Greek and Roman mosaics to the meander family. The analysis and algorithm described in the following have applications to the classification of existing geometric mosaic patterns [19], [30] the design of new patterns [18], and the reconstruction of mosaics that have been partially destroyed by wars or earthquakes.

2. The Roman Pavement in Lewis F. Day's Book

In 1903 Lewis F. Day published a wonderful book on pattern design [6]. The techniques described in his book, as well as the scatterings of wisdom found within its pages, are just as relevant today as they were 100 years ago. In Chapter 17, titled *Patterns Not Strictly Repeating*, we find his Figure 250 that is most relevant to the present discussion, and is reproduced here in Figure 2. Lewis Day has the following to say about this design.

> "The Roman pavement pattern may be described as consisting of a very broad border framing a very small panel. But it may equally well be regarded as a diaper pattern gathered together in places, and finished off at the edges so as to result, more by accident than of set purpose, in a central panel with a broad border, enclosing within it smaller spaces again."

Unfortunately, Lewis Day has nothing more to say about this intricate design, and does not even indicate where this pattern comes from. He describes it simply as a Roman pavement pattern. However, comparison of the two patterns in Figures 1 and 2 leaves little doubt that, in the absence of other knowledge, the two patterns appear to be identical. The use of the word "appear" is warranted since unfortunately the real mosaic pictured in Figure 1 has a large portion of it missing.

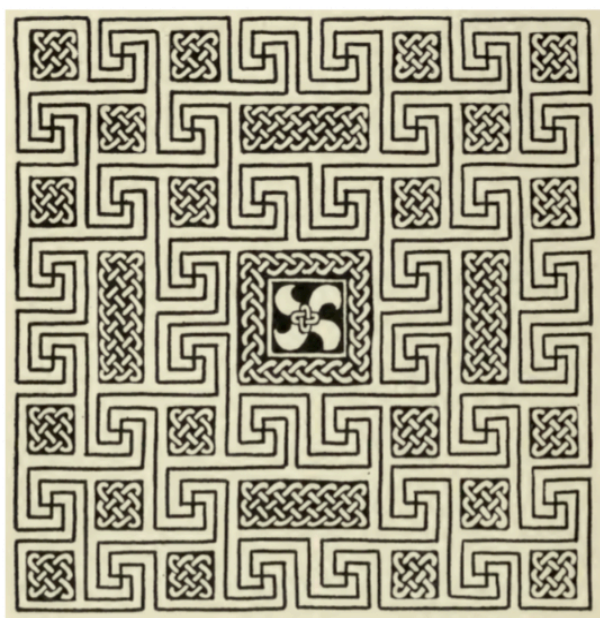

Figure 2: The Roman pavement design from Lewis Day's book [6].

A natural question that arises is whether Lewis Day reconstructed the missing parts evident in Figure 1, and if not, did he take or infer the pattern from somewhere else? To answer this question it helps to distinguish between the collection of panels, or "spaces" as Lewis Day calls them, and the swastika meander pattern that navigates in between these spaces. Figure 3 illustrates the arrangement of the panels in the pattern. There are three types of panels: the one large square in the middle, the four rectangles above, below, to the right, and to the left of the large square, and the twenty small squares arranged in four groups of five located in each corner of the entire region.

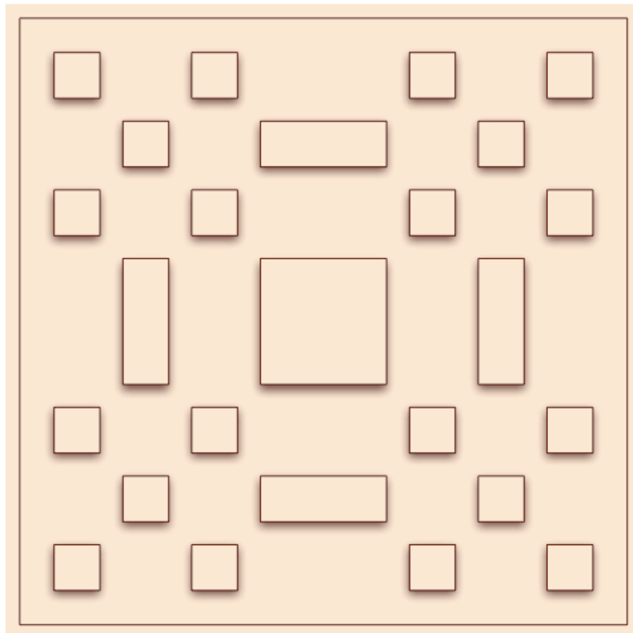

Figure 3: The arrangement of panels in Lewis Day's figure.

Not far from Chedworth there is another Roman villa in Woodchester, that is famous for a mosaic titled *Orpheus and the Beasts*, and thus called the Orpheus mosaic for short [6]. This is a sizable mosaic measuring about 14 meters square. It consists of a large circular panel in the center that depicts various animals. This center panel is surrounded by a collection of geometric patterns. Interestingly, there are four geometric patterns, one in each corner, that consist of exactly the same arrangement of panels shown in Figure 3. Furthermore, the swastika meander patterns in two diametrically opposed corners are exactly the same as the meander pattern in Lewis Day's book (Figure 2 here). The other two corners contain meander patterns that are mirror images of the pattern in Figure 2. In other words in two corners the swastikas open in a clockwise orientation, and in the other two corners they open in a counter-clockwise manner. However, the patterns contained in the interiors of the panels in the Woodchester mosaic are different from and simpler than those present in the Chedworth mosaic. Furthermore, since the patterns in the interiors of the panels are the same in Figures 1 and 2, it is fair to conclude based on this knowledge alone that the swastika meander pattern of Figure 2 is some kind of standard pattern used with the arrangement of panels of Figure 3, and that the pattern in Lewis Day's book is intended to depict the Chedworth mosaic. Most probably Lewis Day "reconstructed" the missing portions of the swastika meander pattern of the Chedworth mosaic by observing the one intact pattern contained in the Woodchester mosaic.

3. The Patterns in the Panels

The three types of panels in the Chedworth mosaic contain four different patterns illustrated in Figure 4. The leftmost image lies at the center of the Chedworth mosaic. It is made up of the ubiquitous Solomon knot [7] consisting of two interlocked strands colored red and green, superimposed on a curved clockwise swastika pattern colored in blue. The Solomon knot is found in many cultures around the world dating back to the distant past. For example it appears on ancient carved stones in Nicaragua [32], on an ancient rock engraving in eastern Angola [12], on Hausa embroidery in Africa [1], as well as on grisaille glass-work of the twelfth-century church at Noirlac, France [33]. This pattern is surrounded by the pattern second from the left, a 3-strand guilloche. Each strand in Figure 4 is colored differently to aid visualization. The 3-strand guilloche is also used in many traditional arts. Not only is it found frequently in Greek and Roman mosaics [7], [15], but also in traditional Persian and Celtic knotwork [1]. The twenty small square panels in the Chedworth mosaic all contain the same pattern, illustrated by the third pattern from the left in Figure 4. It consists of three strands colored red, blue, and green. This 3-strand knot is used frequently in Roman mosaics to decorate square regions. For example it is used in the large square central panels in the four corners of the Orpheus mosaic found in Woodchester [7]. Finally, the rightmost pattern in Figure 4 is contained in the four rectangular panels of the Chedworth mosaic. Examination of this pattern (colored in blue) reveals that it consists of a single closed curve.

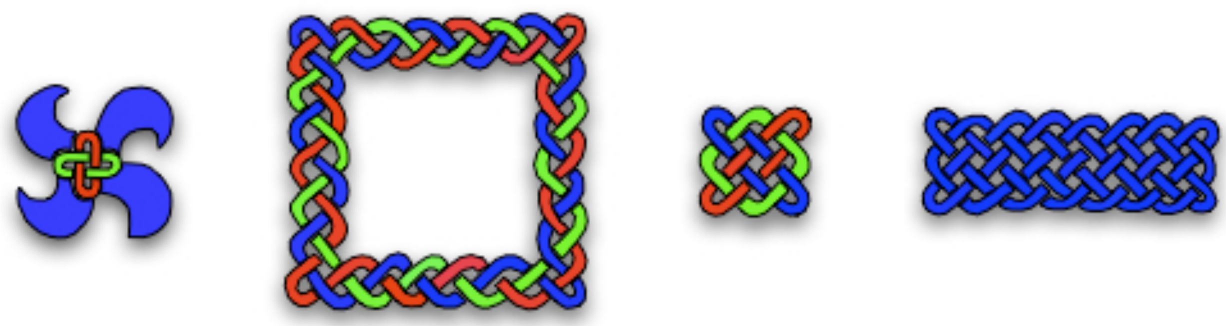

Figure 4: The four patterns contained in the Chedworth mosaic panels.

4. Connections Between Roman Mosaics, *Sona* Drawings, and *Kolam* Patterns

Several cultures from distant parts of the world enjoy traditional visual art practices that unite them with a common thread: the use of meandering cyclic geometric curves that satisfy certain geometric properties [22]. Most prominent among these practices are the *sona* drawing tradition of Angola in Africa [12], and the *kolam* artwork of the Tamil culture in Southern India [9]. A typical sona (kolam) drawing consists of one or more curves that meander around a group of dots previously arranged in a highly symmetrical manner. The pen or other tool used (finger in the case of sand) may not be lifted off the paper during the execution of each closed curve in the drawing, and it may not go over lines already drawn, except to cross over them. Furthermore, such crossings are usually at right angles, or almost so. For some drawings more than one curve is drawn, but the most desirable drawings usually consist of one single curve. Drawings that consist of one curve are called *monolinear* [17], whereas those made up of several curves are referred to as *multilinear* or *polylinear* [12]. When the drawings include the over-under information at their crossings, as in Figure 4, they are called *knot-diagrams* in knot theory, and if they are monolinear they are referred to as *unknots* [28]. Another common requirement of sona drawings is that when a drawing is finished every bounded region must contain exactly one dot in its interior. For a given fixed set of dots many topologically different sona drawings that satisfy these constraints are possible. For example, Figure 5 illustrates three such monolinear topologically different sona drawings on the same set of five dots. It is worth noting that the term 'sona' drawings is used in the more general geometric sense rather than the more restrictive cultural sense. Thus the third drawing in Figure 5 is a traditional sona drawing, but the first two are not. The leftmost drawing is a smoother congruent adaptation of a drawing that appears in Figure 1 of Liu [16]. Nevertheless they are all *geometric* sona drawings. More relevant to the Chedworth mosaic patterns, these 5 dots also admit a (geometric) sona drawing with *two* curves, as shown in Figure 6.

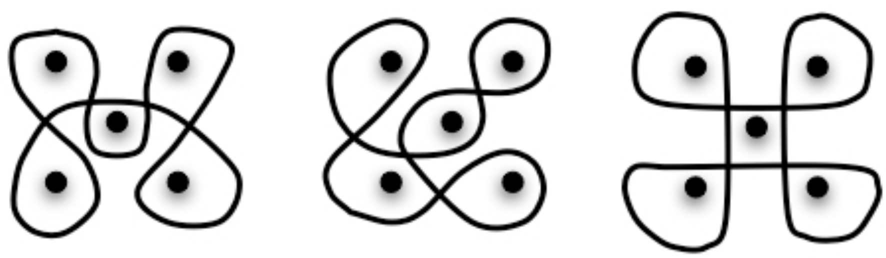

Figure 5: Three topologically different sona drawings on the same arrangement of 5 dots.

Figure 6: A two-strand (red and green) sona drawing on the same arrangement of 5 dots.

The resemblance between the two-strand sona drawing in Figure 6, and the two-strand Solomon knot in Figure 4 is obvious. The Solomon knot does not have the dots that are present in the sona drawing, and the sona drawing does not possess the over-under three-dimensional aspect of the strands that make up the Solomon knot. Other than that, the two curves share the same structure. However, the relationship between the sona drawings and the Roman mosaic meander patterns is even closer than this. Although Lewis Day's drawing of the Chedworth mosaic depicts the designs contained in the 20 small square panels as only 3-strand knots, examination of the photograph in Figure 1 reveals that the strands actually meander around a group of dots just as in the sona drawings. These dots, colored white, are clearly visible in the detail enlargement of the Chedworth mosaic shown in Figure 7. Furthermore, some African designs used in textiles contain sona drawings with the same over-under property prevalent in Roman mosaic meander patterns and Celtic knotwork. Zaslavsky [34] has documented several such examples with photographs: Figure 8.4 (a carved Yoruba calabash from Nigeria), Figures 10.6 and 14.7 (Kuba embroidered raffia cloth from Zaire), and Figure 17.1 (Yoruba beaded boots from Nigeria). Furthermore, the example from Figure 14.7 contains two copies of the three-strand meander found in the small square panels of the Chedworth mosaic.

Figure 7: Detail of Chedworth mosaic showing the white dots in the meander patterns.

The 20 meander patterns contained in the small square panels consist of 3-strand knots that meander around 13 points laid out in a regular 3 by 3 grid. The sona version of this pattern (without the over-under aspect) is shown on the left diagram of Figure 8 using three colors to distinguish between the three strands.

Out of the extremely large number of possible sona drawings that 13 points admit, one may wonder how the Romans arrived at this particular and favorite pattern. The answer is probably a simple algorithm illustrated on the right diagram of Figure 8. First construct a square that encloses all the dots such that the distance between the square and the dots is half the distance between two horizontally adjacent dots. Now imagine that this square is either a billiard table, or made up of mirrors. To construct the blue curve start a billiard ball (or beam of light in case of mirrors) rolling from a point directly on the square and above the upper leftmost dot at an angle of 45 degrees. Then just follow the path of the ball until it returns to the starting point, remembering that (like light) whenever the ball hits an edge of the square it bounces (or reflects) at an angle of 45 degrees, thus turning by and angle of 90 degrees. To trace out the remaining curves, repeat this procedure starting on all the points directly above the dots contained in the top row of dots. This procedure for generating this class of sona drawings is described by Gerdes who calls them *mirror curves* for obvious reasons [12]. In this book Gerdes traces the origin of these algorithms to the practice in Africa, and many other parts of the world, of weaving rectangular mats by folding the strands at 45-degree angles at the sides of the rectangle. Once the final pattern is obtained (Figure 8, right) the reflecting square is removed, the corners of the three curves may be smoothed, and the over-under pattern may be applied to obtain the pattern in Figure 7. In his 1999 book *Geometry from Africa,* Gerdes gives further details about the more general concept of mirror curves [11].

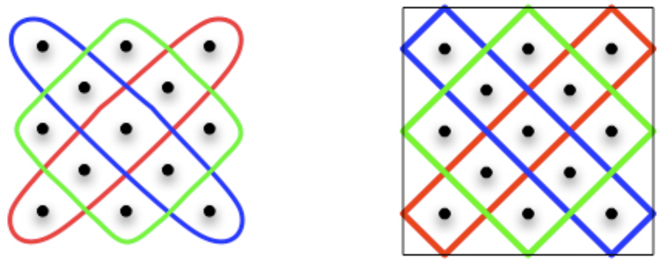

Figure 8: Three-strand meander pattern about regularly spaced points in a 3 by 3 square.

The rightmost pattern in Figure 4 contained in the four rectangular panels of the Chedworth mosaic (also visible in Figure 7) may be constructed with the same algorithm, as illustrated in Figure 9. However, in this case, only one curve is needed, for when one starts the curve, it continues to reflect from the boundary until it completes the entire drawing before returning to the starting point (hence only one color). This happens because the outer rectangle has dimensions 3 by 8, two numbers that are relatively prime.

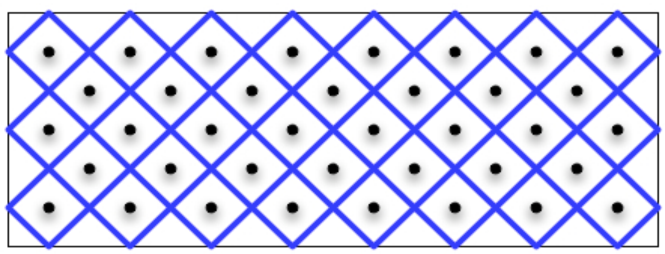

Figure 9: The one-strand meander pattern about regularly spaced points in a 3 by 8 rectangle.

5. Deciphering the Swastika-Meander Pattern

Let us now turn to the analysis of the most intricate and fascinating pattern of the Chedworth mosaic: the rectilinear swastika meander pattern that intertwines the entire space between the panels. This pattern may be viewed as constructed using an 8 by 8 square virtual checkerboard as an underlying guide. In each row and column of the checkerboard there are 4 swastika patterns, for a total of 32. These 32 swastikas are connected by the rectilinear meandering curves. The first question that arises is: how many closed curves make up this pattern? A simple tracing of the curves with different colors reveals that the pattern is composed of 4 different closed curves, as illustrated in Figure 10. They are colored blue, red, green, and black. The black curve stays near the center of the figure, tightly surrounding the large central square panel. The blue curve connects the four outer corners of the figure with the center. The red and green curves connect the centers of the sides of the outer square. To get a clearer picture of how each curve meanders through the space between the panels, each curve is shown by itself in Figure 11.

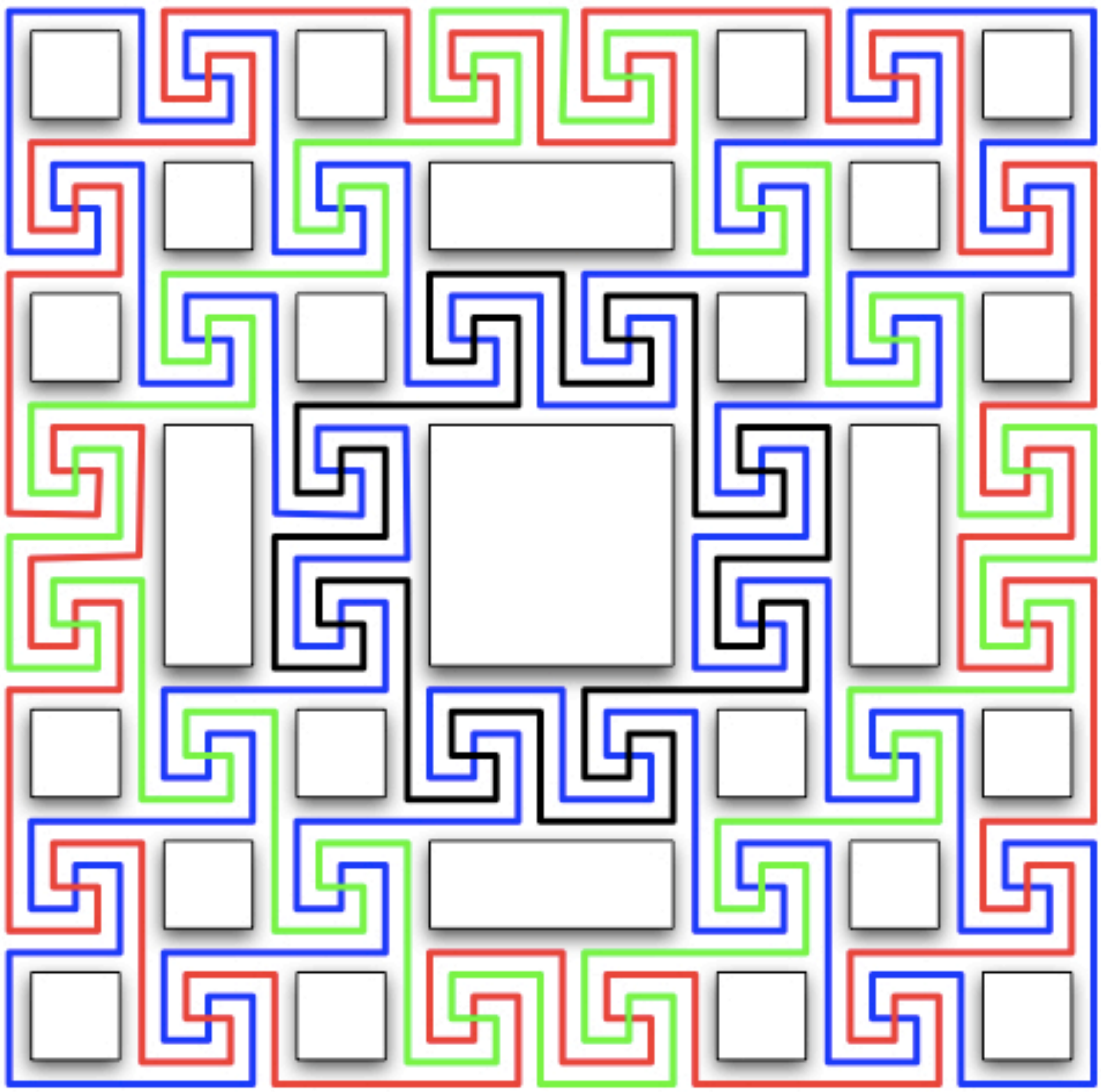

Figure 10: The four curves making up the swastika meander pattern.

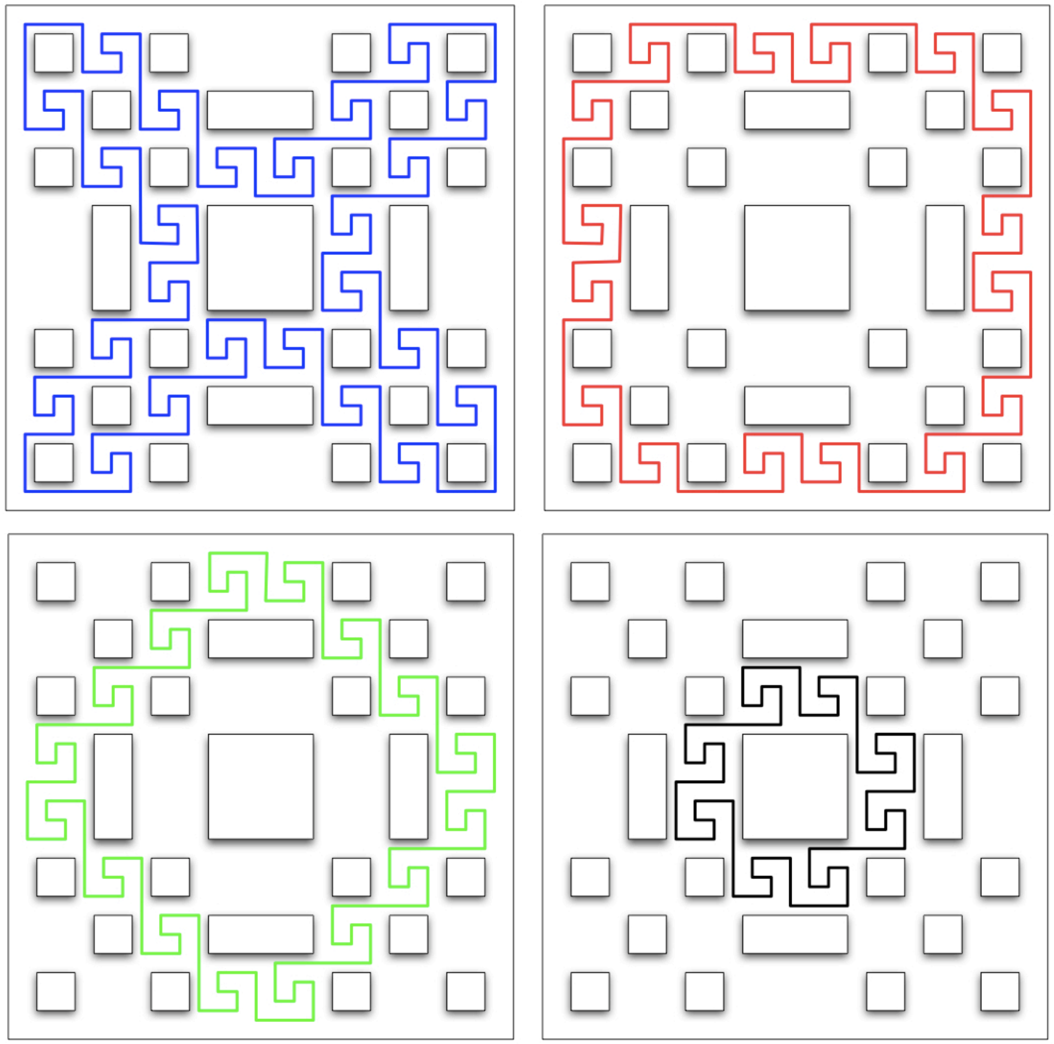

Figure 11: The four monolinear curves making up the swastika meander pattern.

The swastika meander pattern raises other questions about the general Roman design principle at work in this and other similar pavement mosaics found throughout the Roman Empire. For example, all the swastikas in this design open in a clockwise turning manner. That they can all be made to open turning in a counterclockwise manner is obvious, since one merely has to turn over the pattern to obtain its mirror image. Indeed, as pointed out earlier, such counterclockwise versions of this pattern are evident in the *Orpheus* mosaic at Woodchester [7]. The more interesting question is whether a design is possible in which some of the swastikas open in a clockwise manner and others in a counterclockwise manner. The answer to this question is in the affirmative. Indeed, for each of the 24 swastikas we can select, a priori, its required orientation, and then construct a meander pattern that realizes all such choices. To answer this and other questions we outline in the following, a simple algorithm for constructing the Chedworth meander pattern, which we hypothesize was used by the Roman designers at Chedworth and elsewhere. The algorithm begins identically to the one we already used to construct the pattern in Figure 8. As before, we construct squares that surround the entire pattern that act as mirrors that reflect light or act as the edges of a billiards table (see Figure 12, and note that there are two squares, the larger one for the two outer curves, and the smaller one for the inner curve). The 25 small square panels will play the role of the dots around which the curves meander. The difference now is that the large square panel in the middle, as well as the four rectangular panels that surround it, *also* act as reflecting mirrors or bouncing edges for the billiard balls. To begin our construction we draw four purely reflecting curves as shown in Figure 12. Consider the corner-visiting curve for example. As before, we start tracing the curve from a point above the leftmost square panel of the top row, at an angle of 45 degrees in a southeasterly direction. Whenever the billiard ball encounters the outer square or one of the white panels it is reflected at 45 degrees, until it returns to the starting position. A similar process is used to generate the other two curves. Finally, the black curve in the interior is generated by starting at a midpoint of the interior side of one of the rectangular white panels.

The first part of our algorithm has been suggested before for the construction of sona drawings in the work of Gerdes [10]. It is also the underlying design principle for constructing Celtic knots elaborated in the books by Bain [1] and Meehan [20], and is analyzed in a more general mathematical context by Schlatter [25].

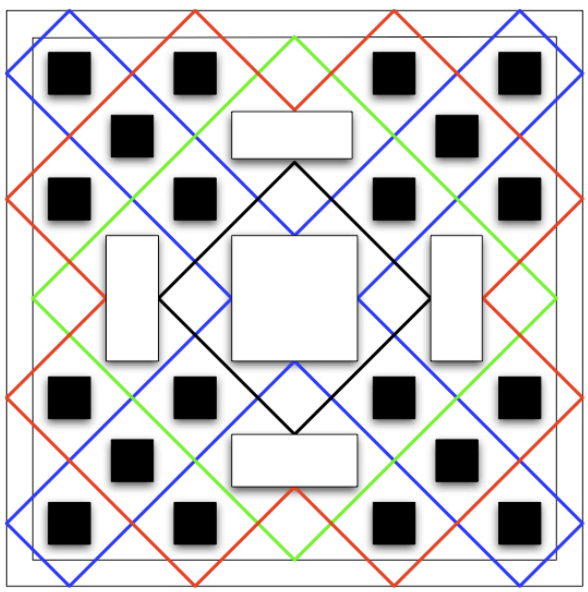

Figure 12: The algorithm for constructing the Chedworth swastika meander pattern.

The reader will notice upon comparison of Figures 12 and 10 that every crossing between two meander curves in Figure 12 corresponds to a position in Figure 10 that contains a swastika pattern. To complete the design, the remaining steps needed to convert the pattern of Figure 12 to that in Figure 10 are straightforward. Note that all the lines used in Figure 12 have a diagonal orientation at 45 degrees. To solve this problem we deform the figure such that all the line segments used are either vertical or horizontal while maintaining the same topology as the drawing that contains diagonal lines.

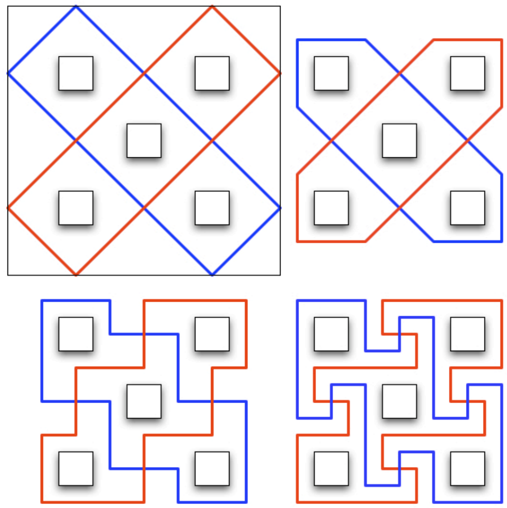

Figure 13: Straightening the curves and twisting the swastikas.

Finally, to convert each crossing to a swastika we merely twist it in the direction desired. Intuitively, the process may be viewed as follows. Imagine the curves are made up of ropes lying on the ground. A crossing between a vertical and horizontal line segment divides the plane into four regions. Imagine placing four fingers of one hand on the ground so that one finger touches the ground in each region. Now twist or rotate the hand in some direction, say clockwise, while dragging the ropes with the fingers, and restricting the ropes to lie on only vertical and horizontal lines. As an example consider the swastika meander pattern found on the geometric mosaic carpet from the *House of the Evil Eye* in Antioch [15], [29]. The process is illustrated in Figures 13 and 14 with each curve separately for clarity, and at the bottom, for both curves together. Note that the intersection point remains fixed during the rotation, and the rotation operation may be carried out indefinitely to obtain deeper and deeper spirals emerging from the swastikas. Note that every time the swastika is twisted it may need to be rescaled so that the separation distances between all the pairs of adjacent parallel lines remain the same.

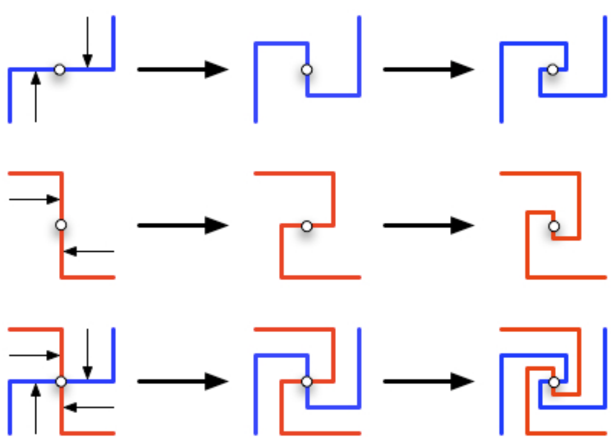

Figure 14: Twisting a crossing to create a swastika spiral pattern.

The algorithm described answers several other questions concerning such designs. Since we are free to twist each crossing in either clockwise or counterclockwise orientations, we are free to choose any such combination. As an example consider the pattern of five dots in the sona drawing of Figure 6 in the context of Roman meander pattern design for mosaics. Figure 13 (lower right) shows a swastika meander version of this pattern with all the swastikas opening in a clockwise direction. On the other hand Figure 15 shows examples with one counter-clockwise swastika (left) and two counter-clockwise swastikas (right). To our knowledge these designs have not been found on any Roman mosaic archaeological sites. Therefore our algorithm may also be used to obtain new designs.

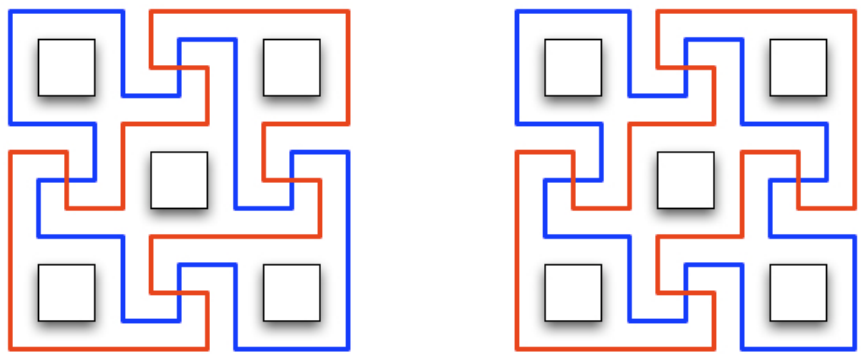

Figure 15: One counter-clockwise swastika on the left and two on the right.

Another question concerns the number of closed meandering curves necessary to complete an overall swastika meander pattern. Notice first that the four curves in Figure 12 may be viewed as sona or kolam drawings in which some of the regions do not contain a small black square panel. These are the regions adjacent to the large central white square panel and the four white rectangular panels. Also, as just pointed out, any sona drawing may be converted to a swastika meander pattern by twisting the crossing points. Furthermore, for any set of dots there are many possible sona drawings consisting of just one curve. Therefore for the Chedworth panel arrangement there exists a very large number of possible swastika meander patterns that make use of only one curve, if we do not require the final drawing to contain all 32 swastikas. If no reflecting panels are used, and the overall region is a square or a rectangle whose sides have a common factor, such as is the case in Figures 12 and 8, then we need more than one curve, whereas if the side lengths have no common factor (i.e., are relatively prime), such as the 3 by 8 panel in Figure 9, then one curve is sufficient. If reflecting panels are used, as in the Chedworth design, then the situation is more complex. If no restrictions are imposed on the placement of the panels then it is always possible to construct a sona meander pattern with a single curve regardless of the dimensions of the rectangle containing the dots. This follows from the work of Chavey [4], who proved several theorems about drawing symmetric meander patterns that depend on the symmetrical arrangements of the reflecting panels used.

6. Conclusion

At first glance the swastika meander pattern found on the dining room floor mosaic at the Roman villa in Chedworth, England, appears to be quite a complex and intricate design that has garnered praise for the intelligence and ingenuity of its Roman designers. The geometric analysis provided here of this pattern reveals a simple algorithm by which the Romans could have easily constructed this and other such seemingly complex patterns. This algorithm permits the construction of new alternate designs. Furthermore, the algorithm may be used to classify Roman meander patterns, and to reconstruct them in partially destroyed mosaics.

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