Scheduling, Map Coloring, and Graph Coloring

Scheduling via Graph Coloring: Final Exam Example

Suppose want to schedule some final exams for CS courses with following course numbers:

1007, 3137, 3157, 3203, 3261, 4115, 4118, 4156

Suppose also that there are no students in common taking the following pairs of courses:

1007-3137

1007-3157, 3137-3157

1007-3203

1007-3261, 3137-3261, 3203-3261

1007-4115, 3137-4115, 3203-4115, 3261-4115

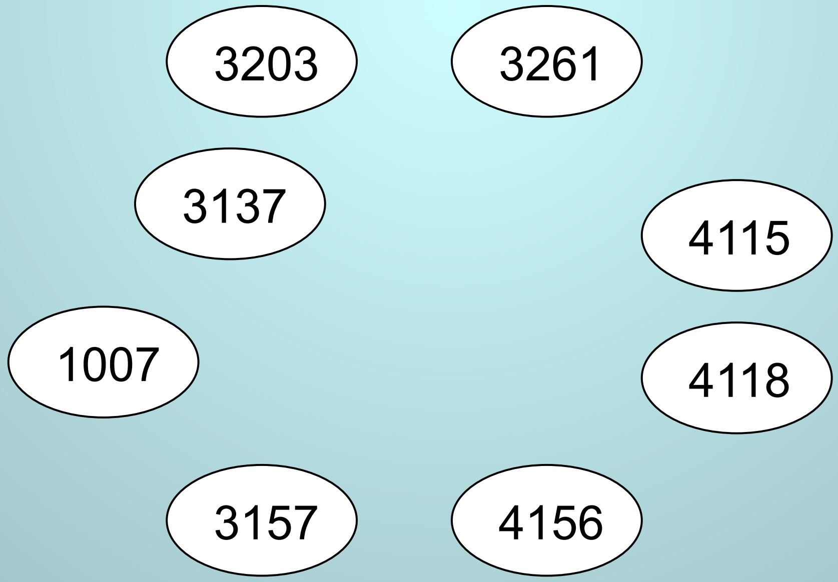
1007-4118, 3137-4118

1007-4156, 3137-4156, 3157-4156

How many exam slots are necessary to schedule exams?

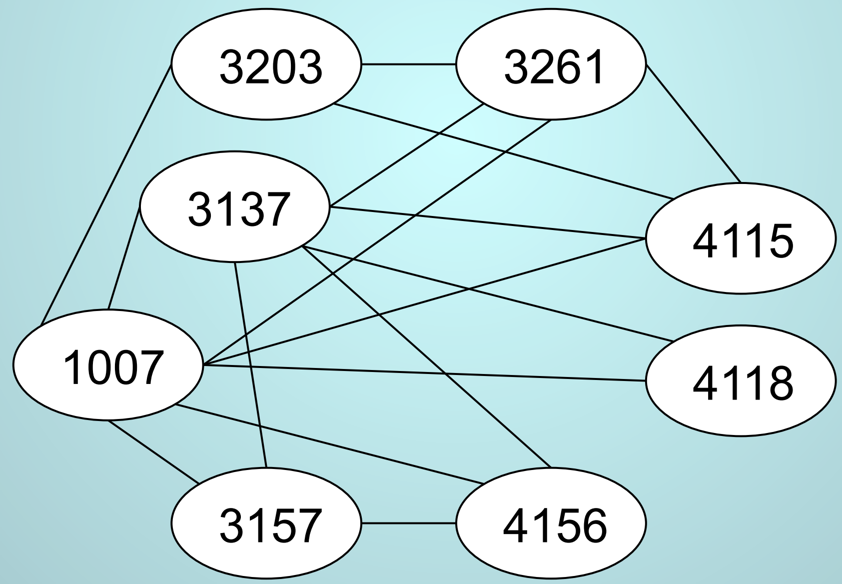
Graph Coloring and Scheduling

- Convert problem into a graph coloring problem.
- Courses are represented by vertices.
- Two vertices are connected with an edge if the corresponding courses have a student in common.



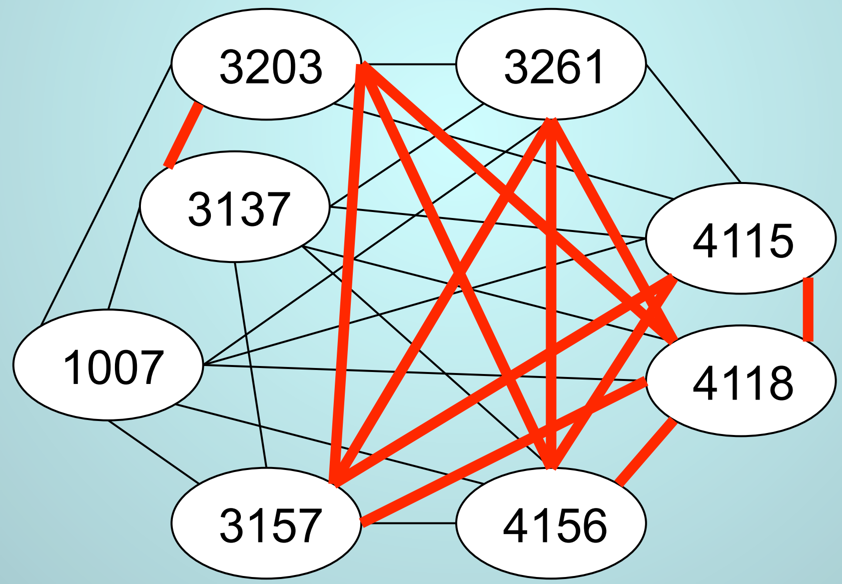
Graph Coloring and Scheduling

One way to do this is to put edges down where students mutually excluded…



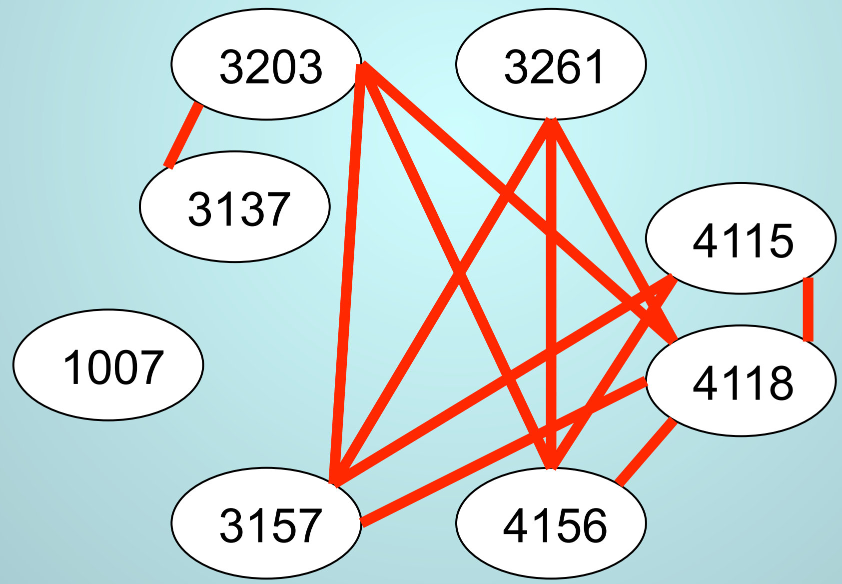
Graph Coloring and Scheduling

…and then compute the complementary graph:



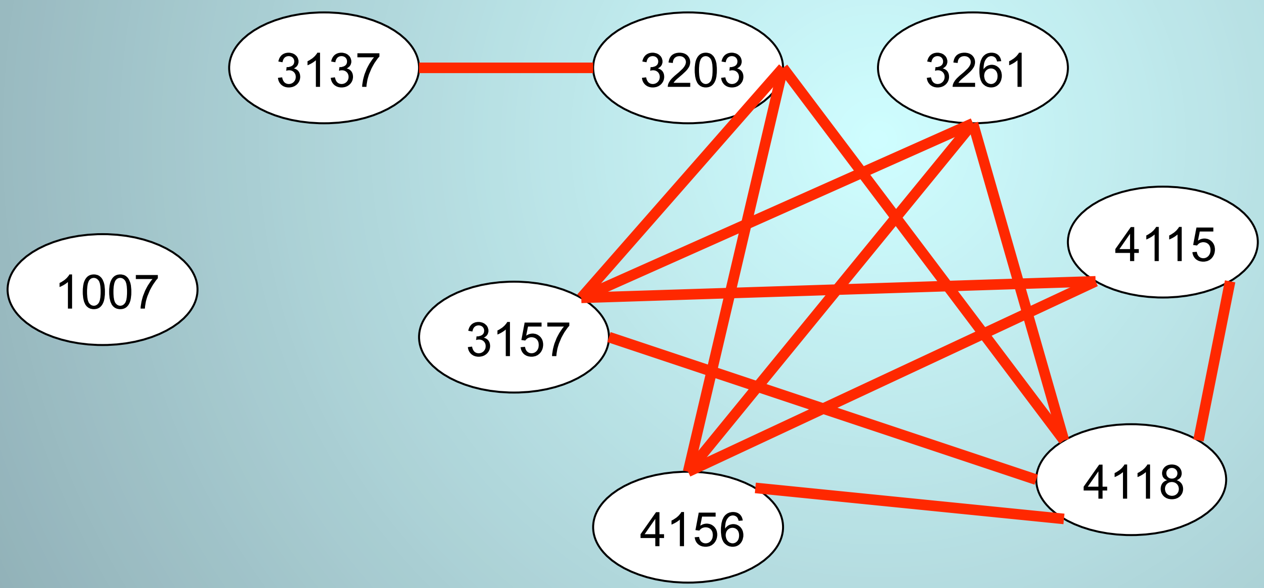
Graph Coloring and Scheduling

…and then compute the complementary graph:



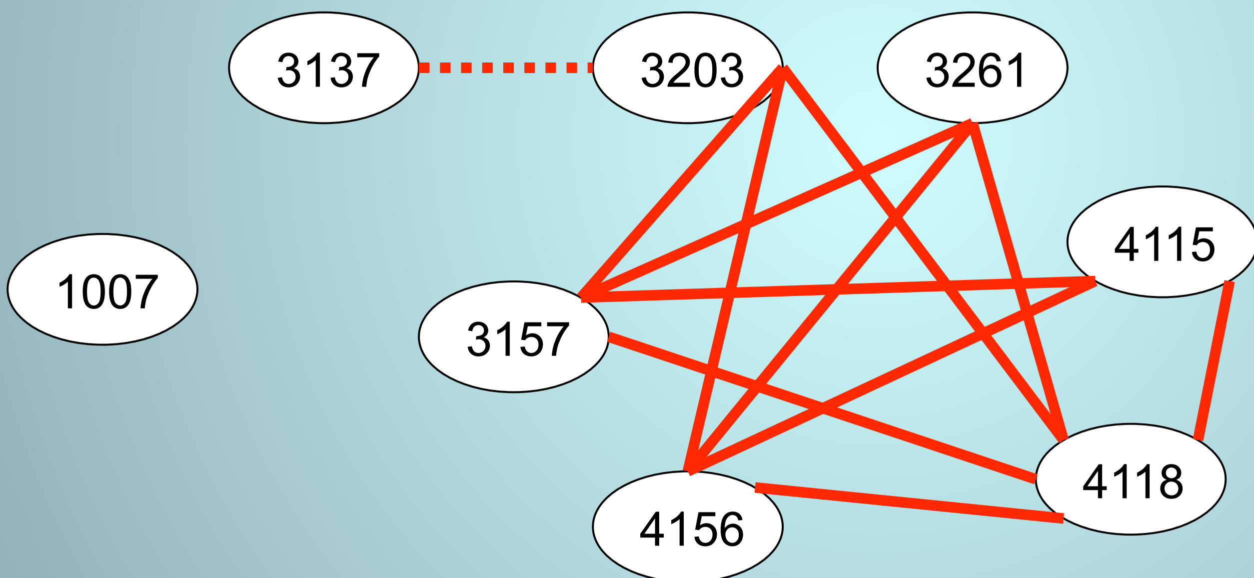
Graph Coloring and Scheduling

Redraw the graph for convenience:



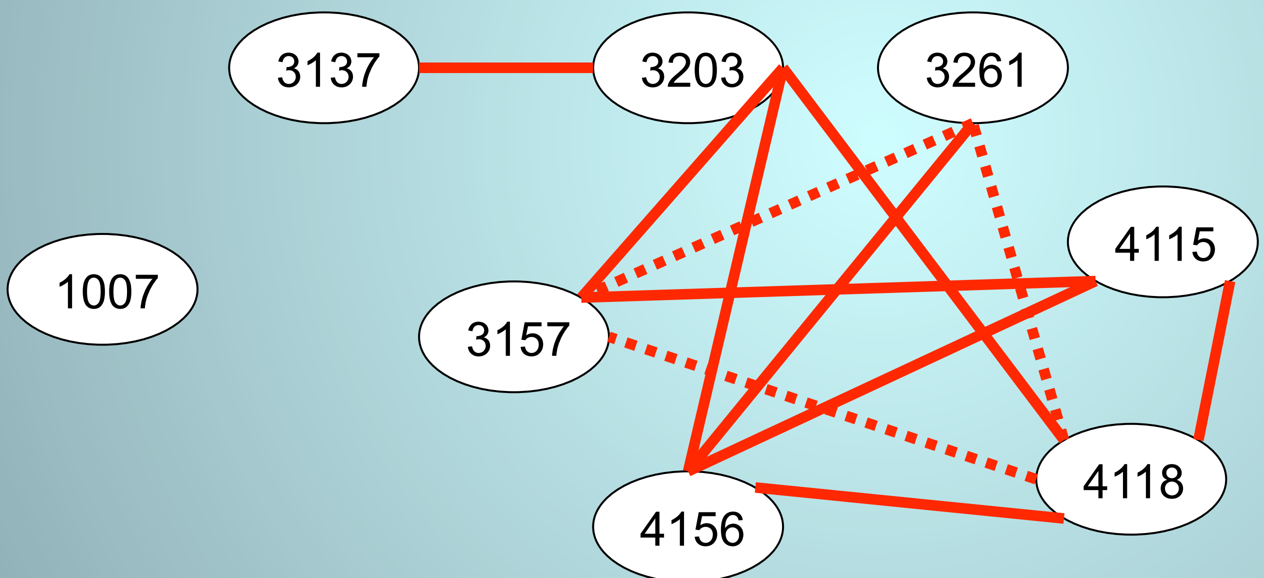
Graph Coloring and Scheduling

The graph is obviously not 1-colorable because there exist edges.



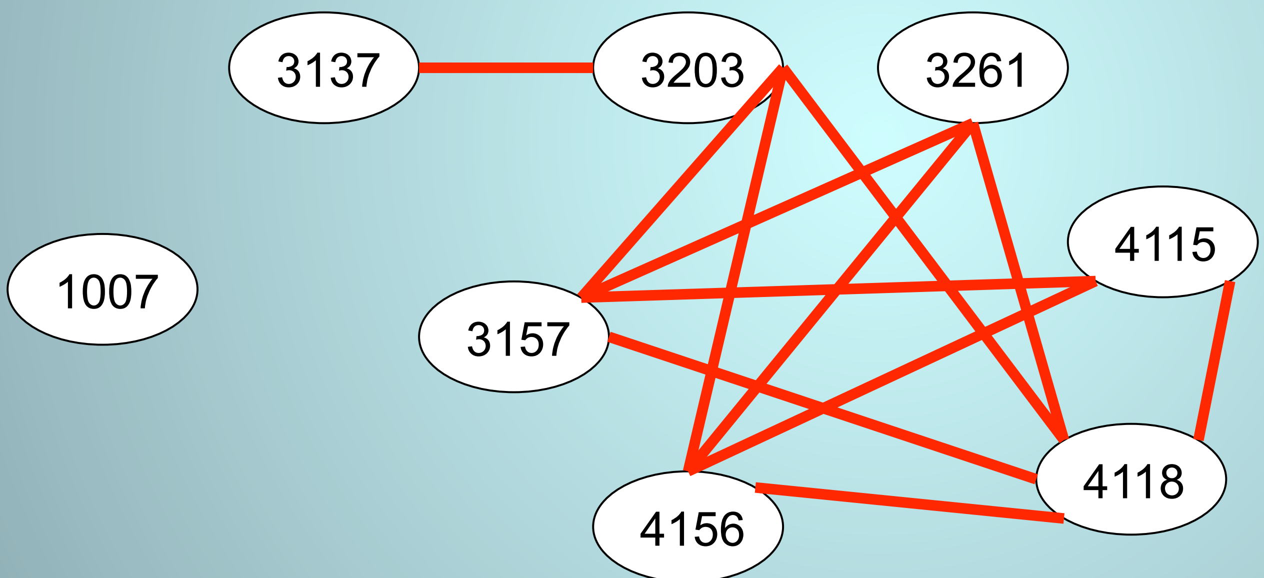
Graph Coloring and Scheduling

The graph is not 2-colorable because there exist triangles.



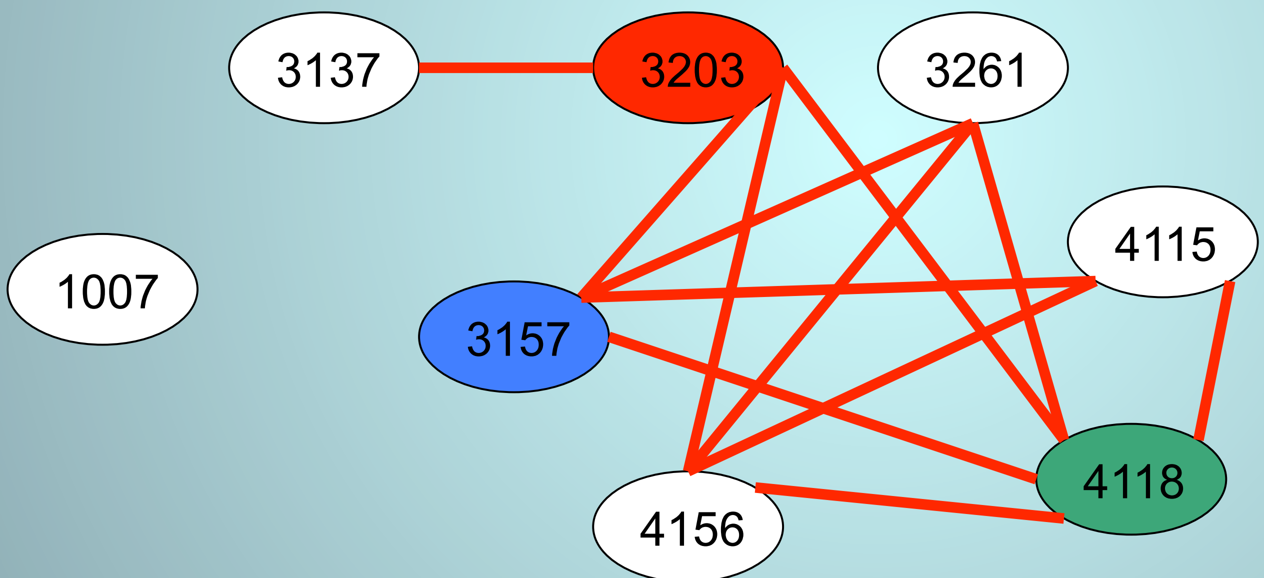
Graph Coloring and Scheduling

Is it 3-colorable? Try to color by Red, Green, Blue.



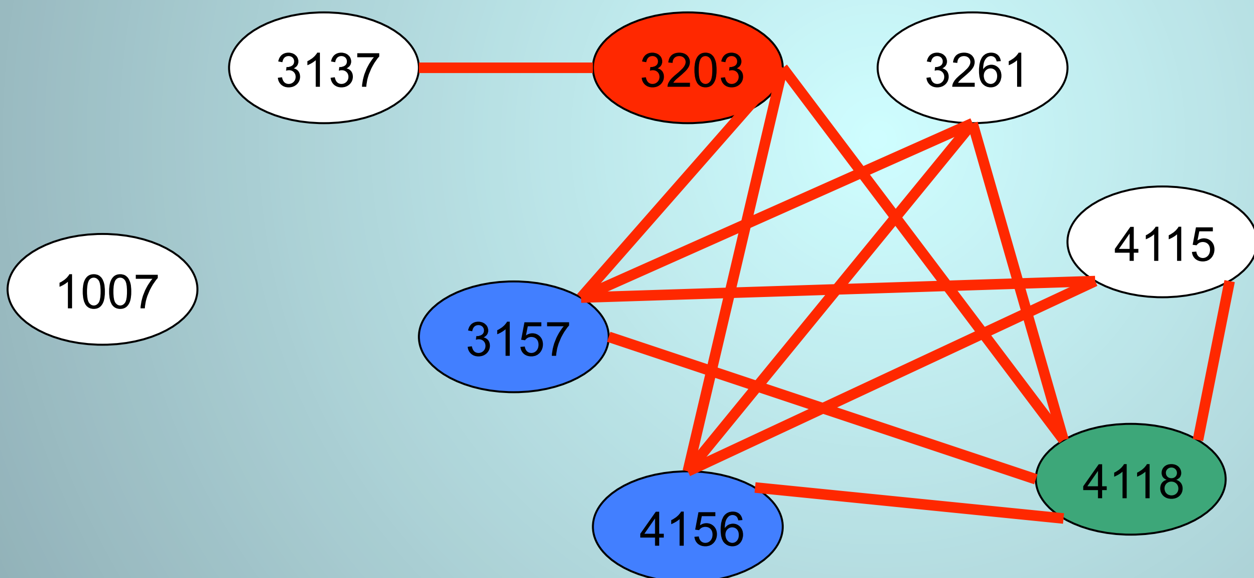
Graph Coloring and Scheduling

Pick a triangle and color the vertices 3203-Red, 3157-Blue and 4118-Green.



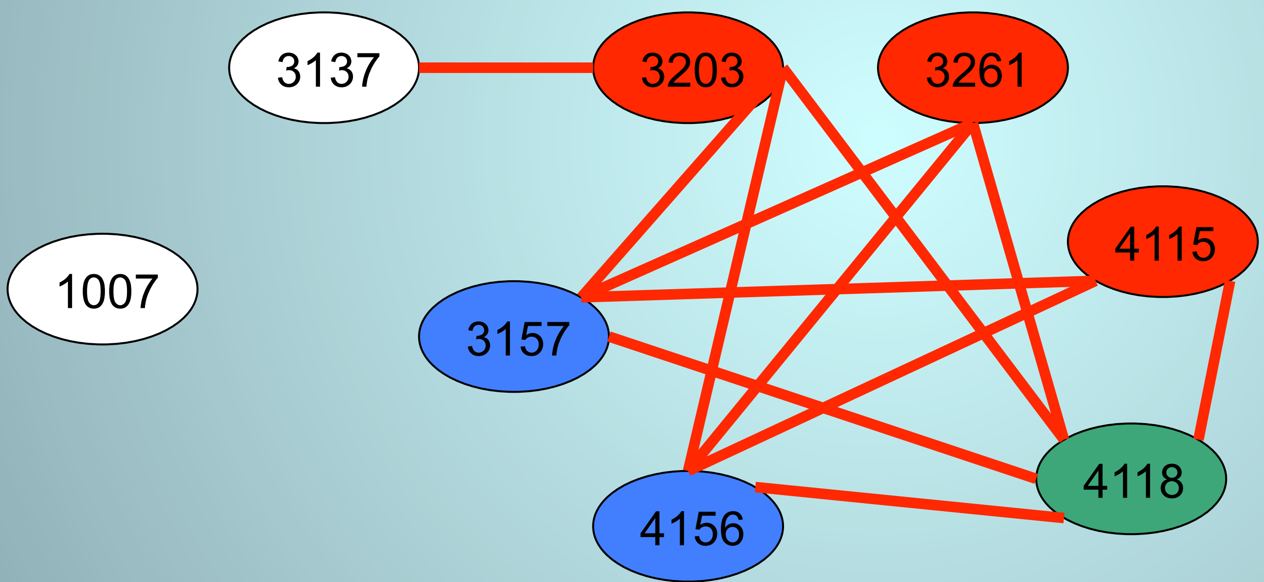
Graph Coloring and Scheduling

So 4156 must be Blue:



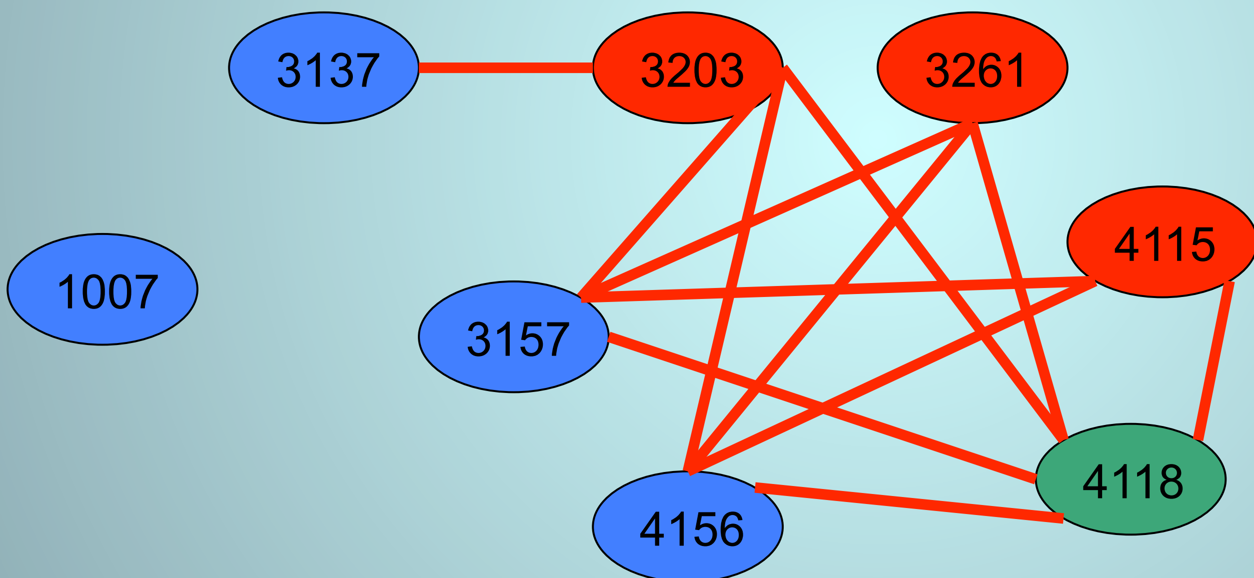
Graph Coloring and Scheduling

So 3261 and 4115 must be Red.



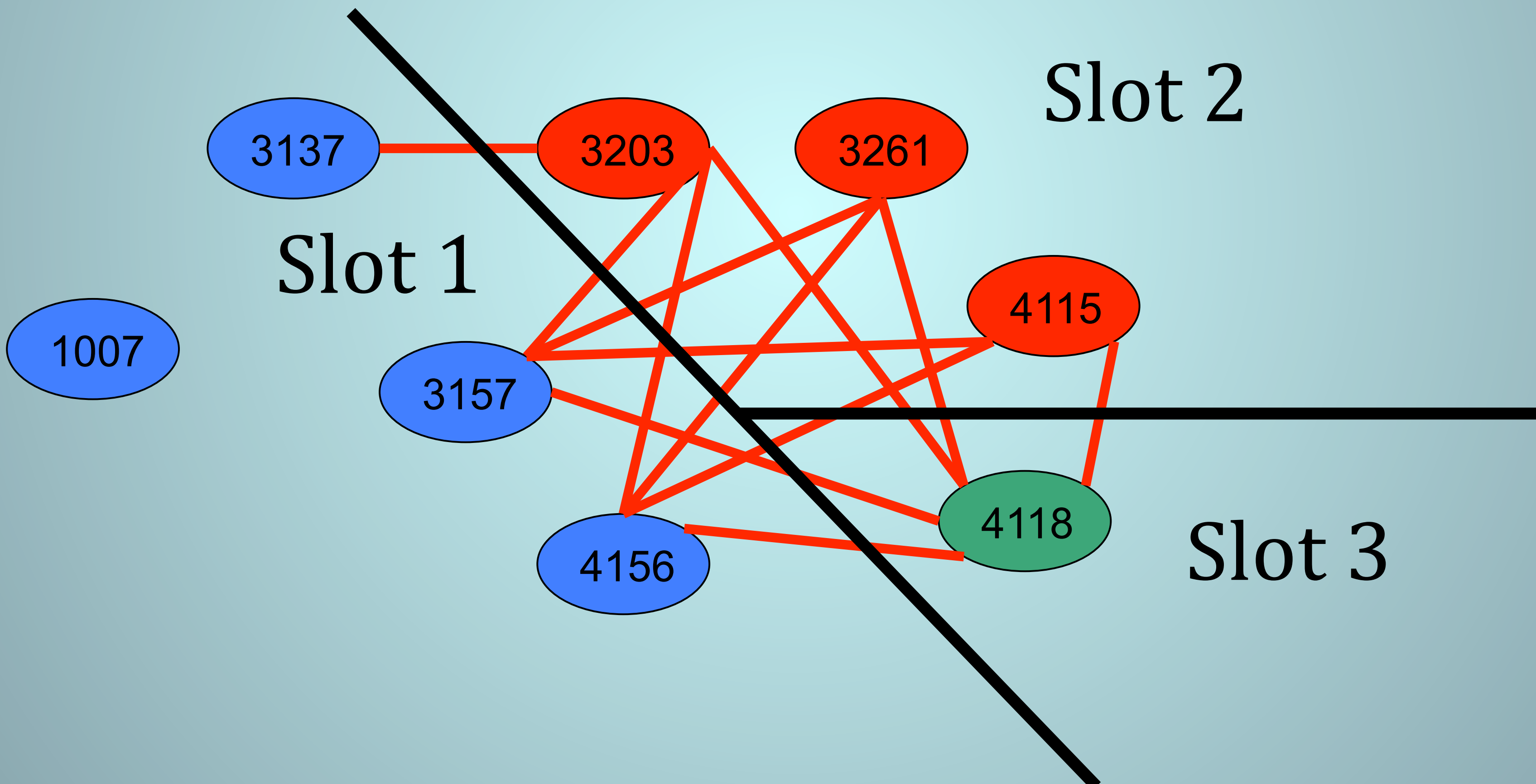
Graph Coloring and Scheduling

3137 and 1007 easy to color – pick Blue.

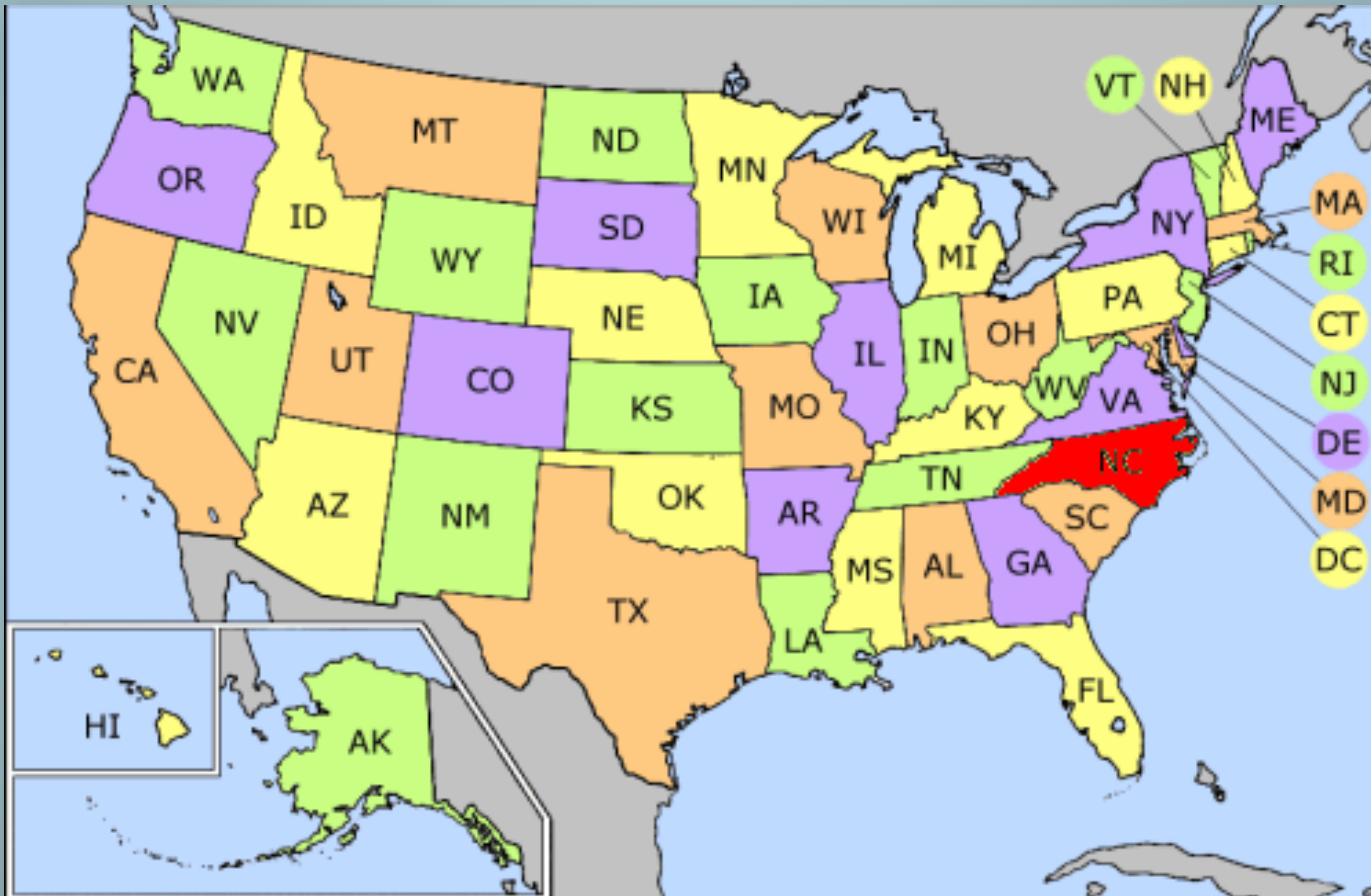


Graph Coloring and Scheduling

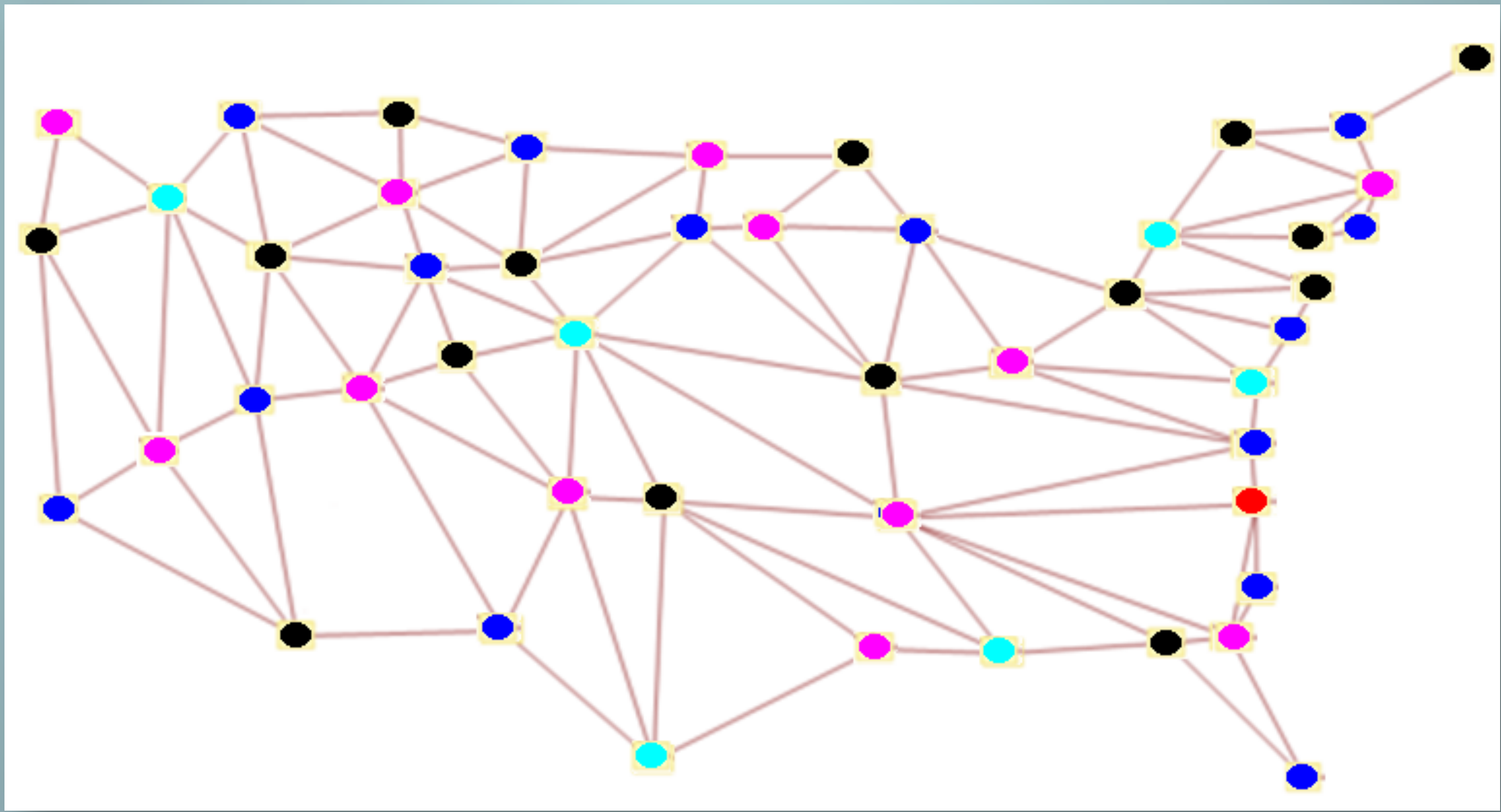
Therefore we need 3 exam slots:



Map Coloring: 5-Coloring the Continental US

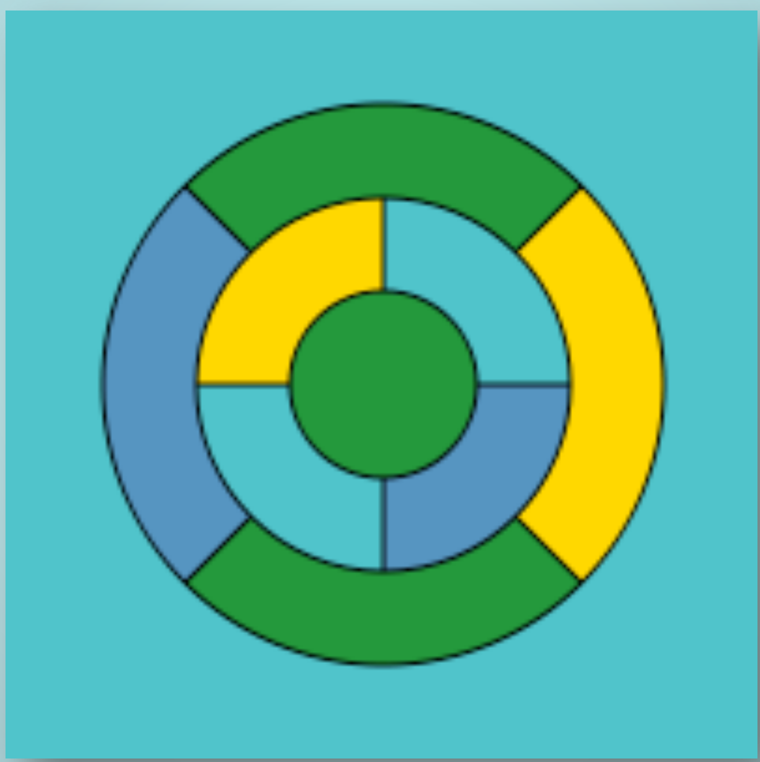


5-Color Vertex Coloring of the Continental US



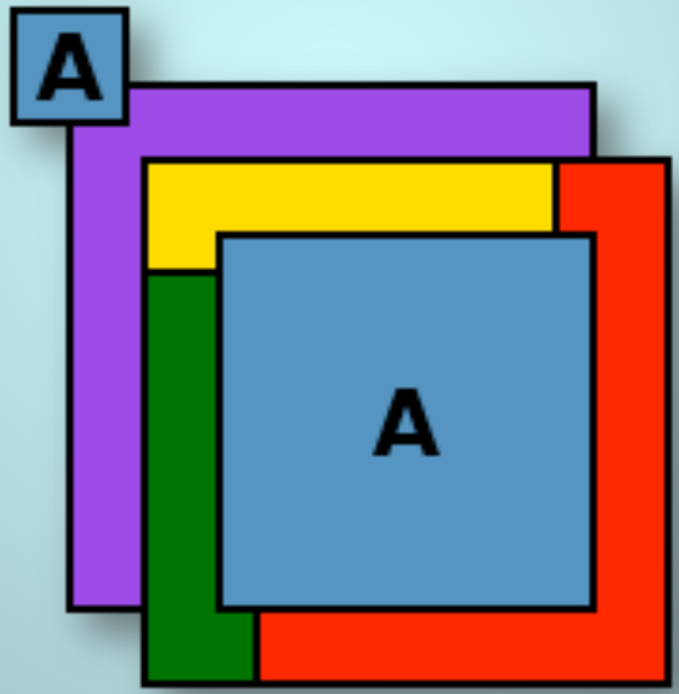
4-Coloring of the Continental US:
4 Colors Suffice for Continuous Planar Maps

Four Colors Suffice

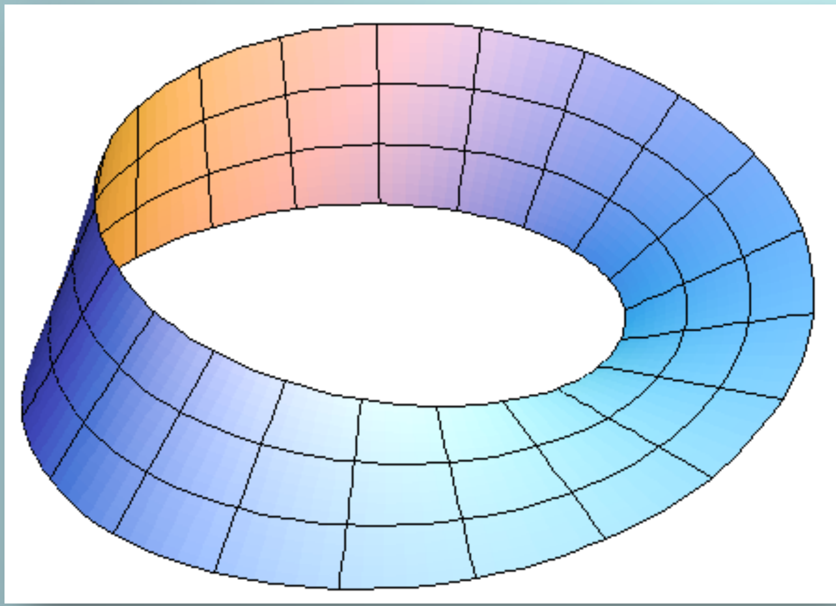


An arbitrary number of colors may be needed if regions are not contiguous.

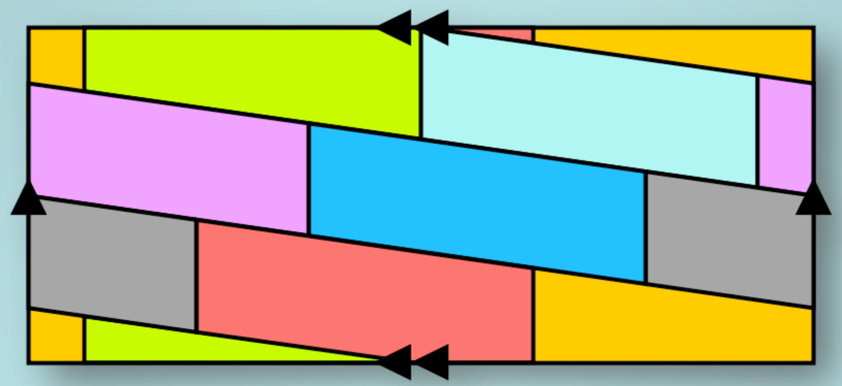
This example needs 5:

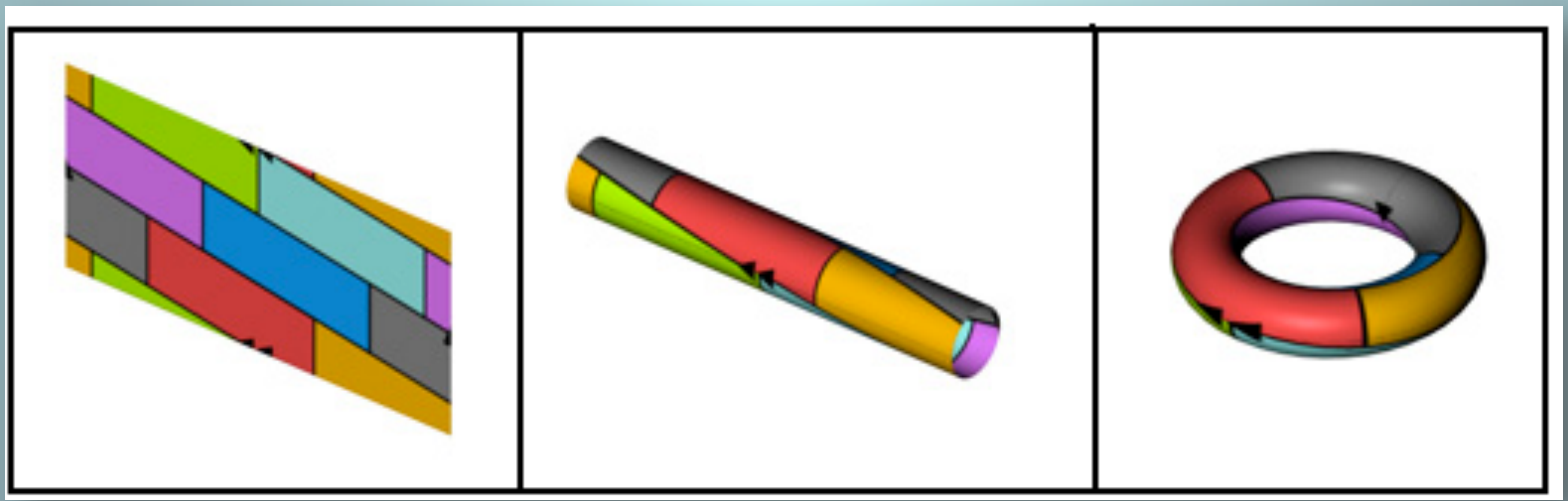


Six colors may be needed if continuous regions lie on a Möbius strip.



Seven colors may be needed if continuous regions lie on a Torus.

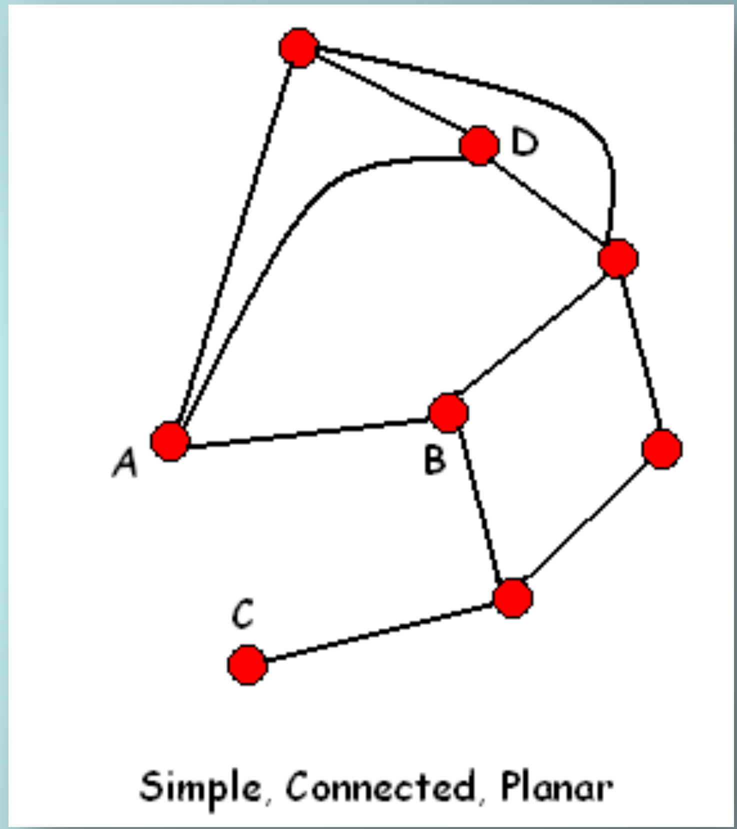




Basic Theorems

- Handshaking Lemma:
- In any graph, the sum of the degrees of the vertices is equal to twice the number of edges.

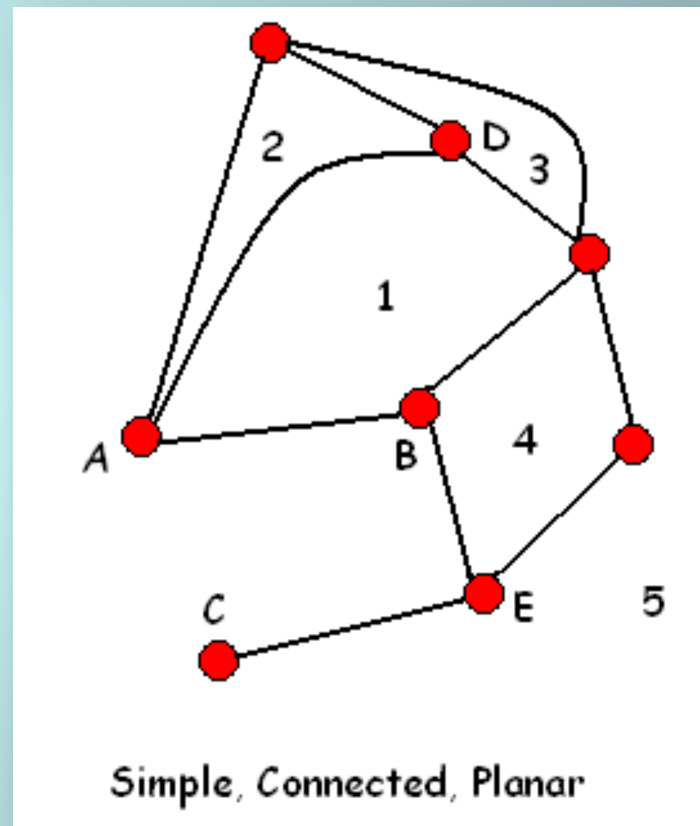
$$\sum_{i=1}^{n} \deg(v_i) = 2E$$



Planar Handshaking Theorem

- In any planar graph, the sum of the degrees of the faces is equal to twice the number of edges.

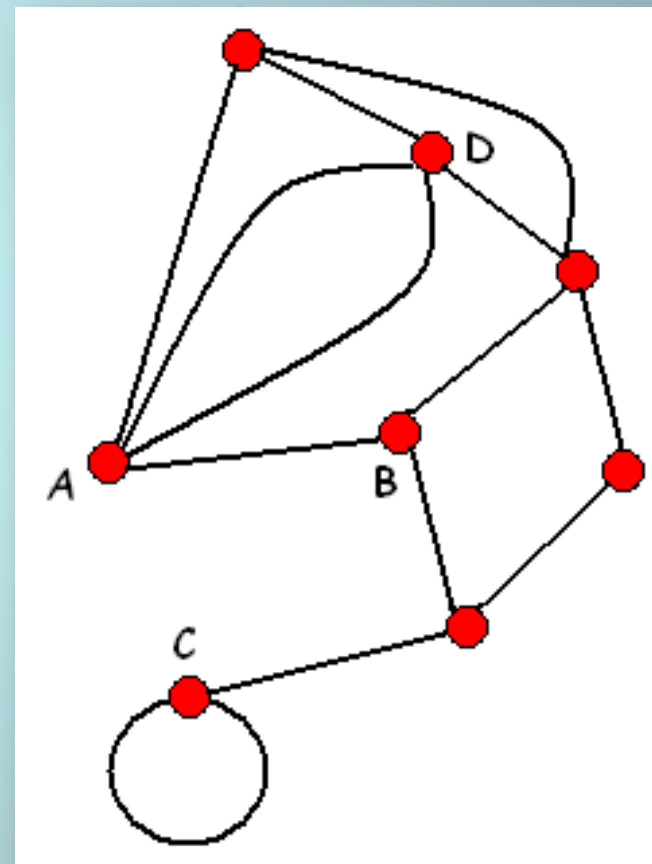
$$\sum_{i=1}^{k} \deg\left(f_k\right) = 2E$$



Euler's Formula

- In any connected planar graph with V vertices, E edges, and F faces,

$$V - E + F = 2.$$



Two Theorems

- Two theorems are important in our approach to the 4-color problem.

- The first puts and upper bound to the number of edges a simple planar graph with V vertices can have.

- The second puts an upper bound on the degree of the vertex of smallest degree.

Theorem 1: Let G be a simple connected planar graph with $V \geq 3$, then $E \leq 3(V-2)$.

Proof: G is a simple connected planar graph, so $\sum \deg(f_i) = 2E$.

Since the graph is simple, all faces must be of degree 3 or more (there are no loops or multiple edges), so $\sum \deg(f_i) \geq 3F$.

Consequently, $F \leq \frac{2}{3}E$.

Also, $V - E + F = 2$, so $V - E + \frac{2}{3}E \geq 2$ and $E \leq 3(V-2)$.

Vertices of degree ≤ 5

Theorem 2: Let *G* be a simple planar connected graph.
Then *G* has at least one vertex of degree 5 or less.

Proof: We proceed by contradiction. Suppose all of the vertices of *G* have degree 6 or more, so $\sum \deg v_i \geq 6V$. But, by the handshaking lemma, $\sum \deg v_i = 2E$. So $E \geq 3V$.
But *G* is a simple planar connected graph, so $E \leq 3(V-2)$.

This contradiction shows it is not possible for all of the vertices to be of degree 6 or more, so at least one must be of degree 5 or less.

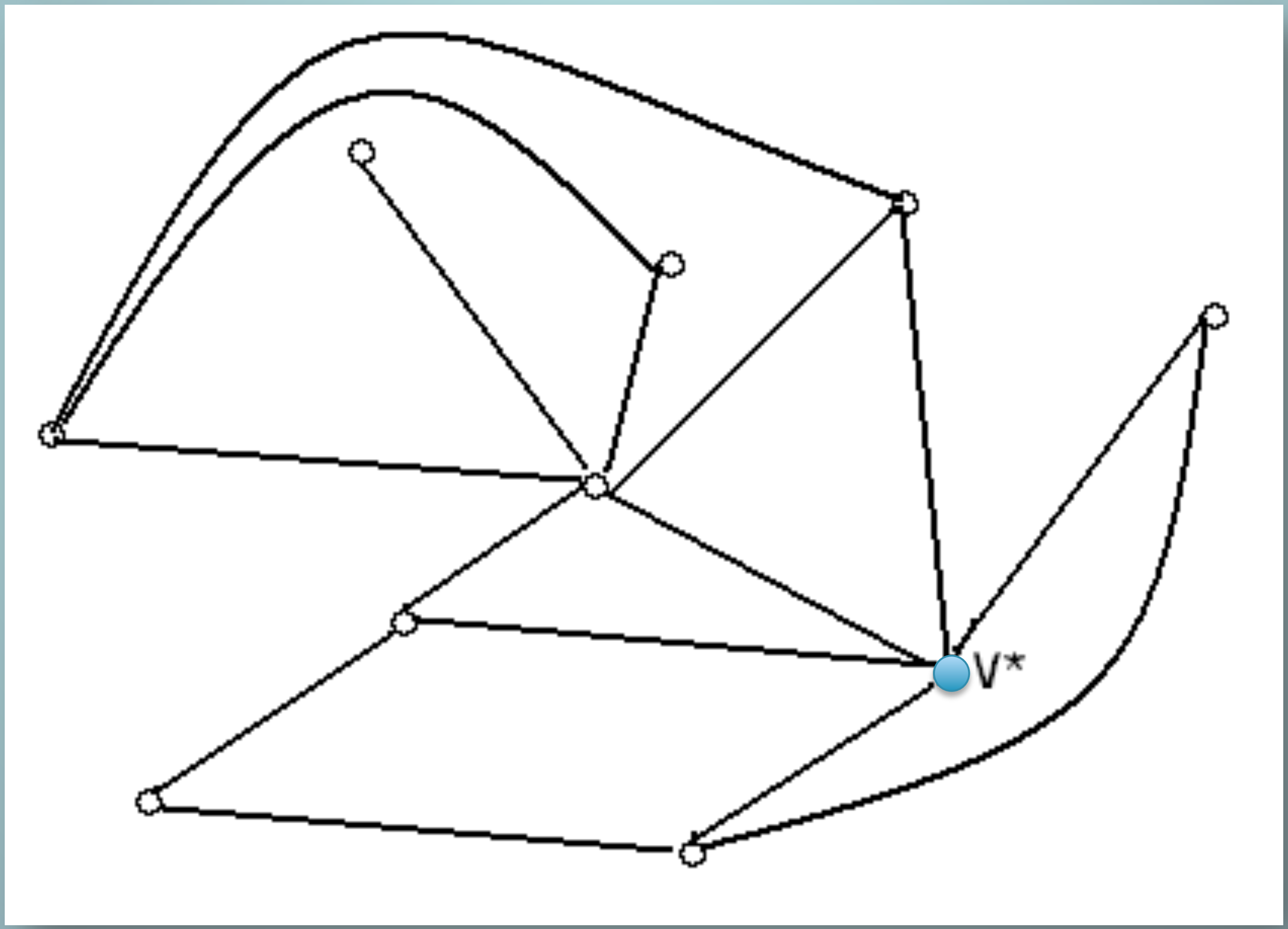
The 6-Color Theorem: Every simple connected planar graph is 6-colorable.

Proof: We proceed by induction. Let P_n be the statement that every connected simple planar graph with *n* vertices is 6-colorable.

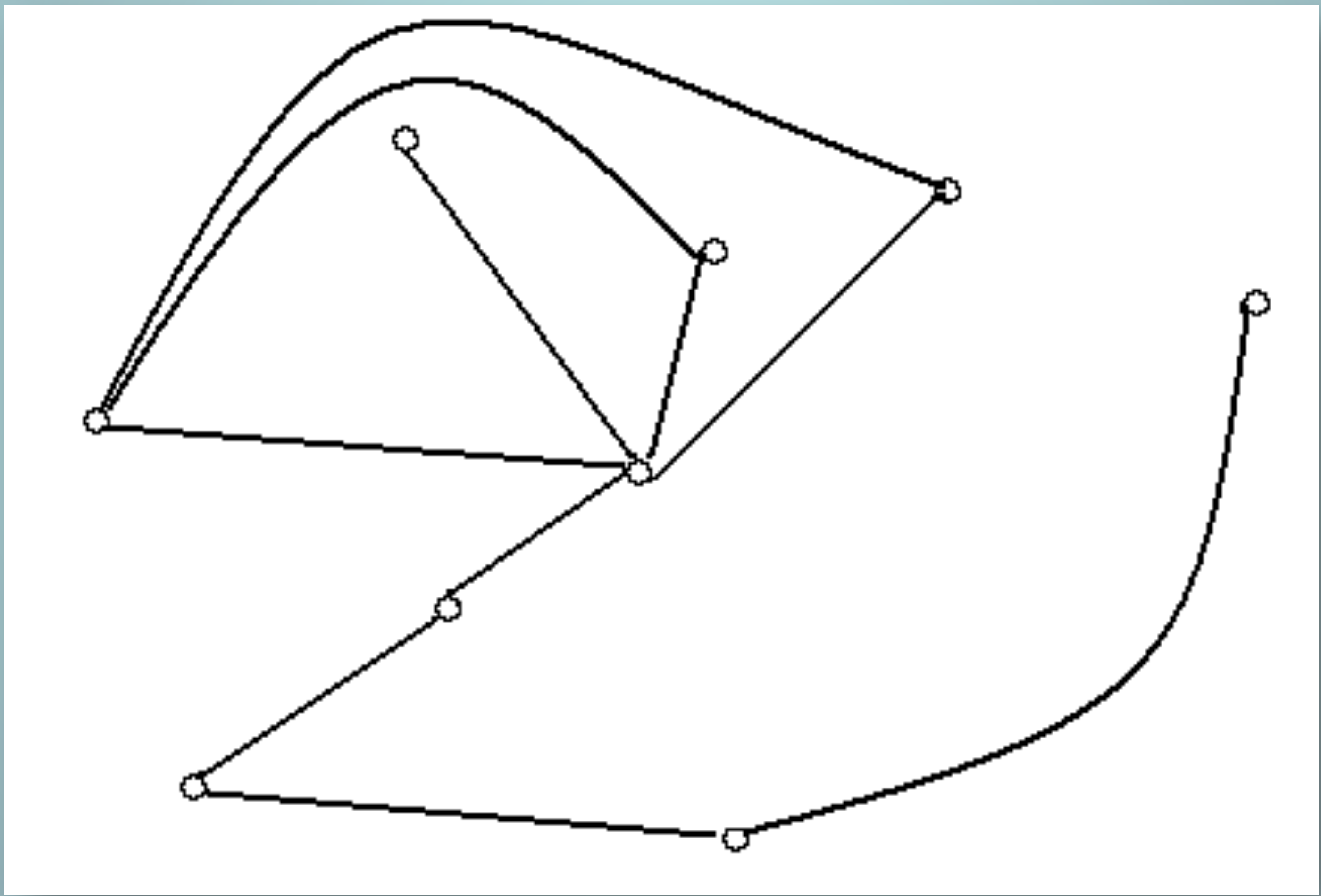
Basis Step: Clearly $P_1, P_2, \cdots, P_6$ are true, since any graph of 6 or fewer vertices can be colored with 6 colors.

Assume every connected simple planar graph with *k* vertices is 6-colorable. We must prove that every connected simple planar graph with $k+1$ vertices is 6-colorable.

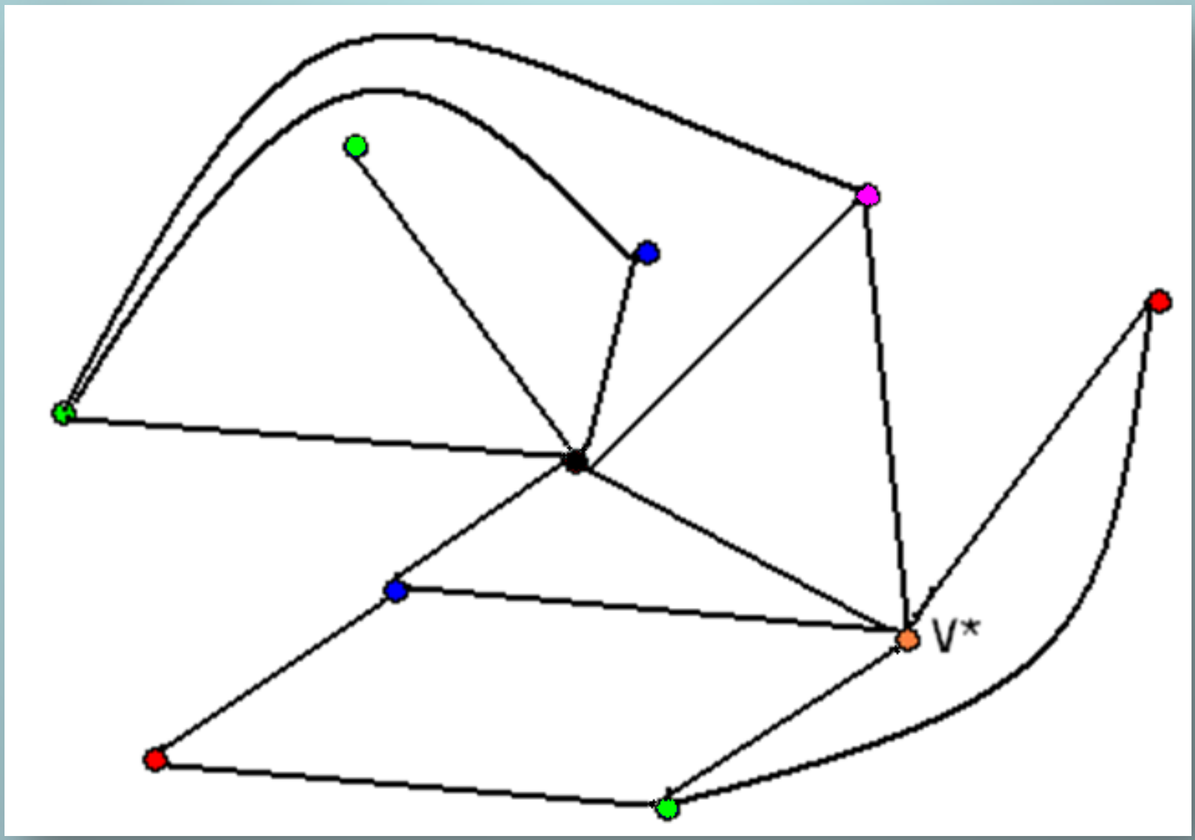
Consider a graph with $(k+1)$ vertices.
Find V^* with degree 5 or less



Remove *V** and all incident edges. The resulting subgraph has *k* vertices. Therefore it can be 5-colored.



Replace V* and incident edges. Since we have 6 colors available and at most 5 adjacent vertices, use the remaining color for V*.



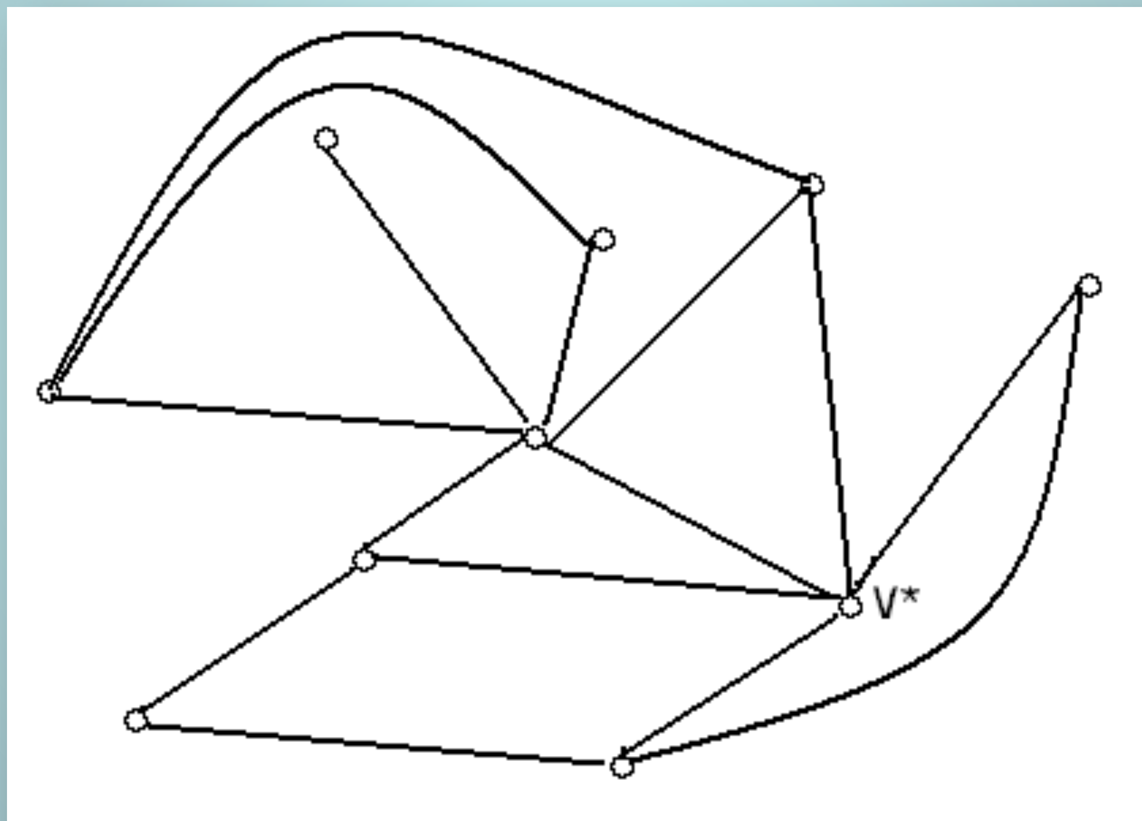
The 5-Color Theorem:

All connected simple planar graphs are 5 colorable.

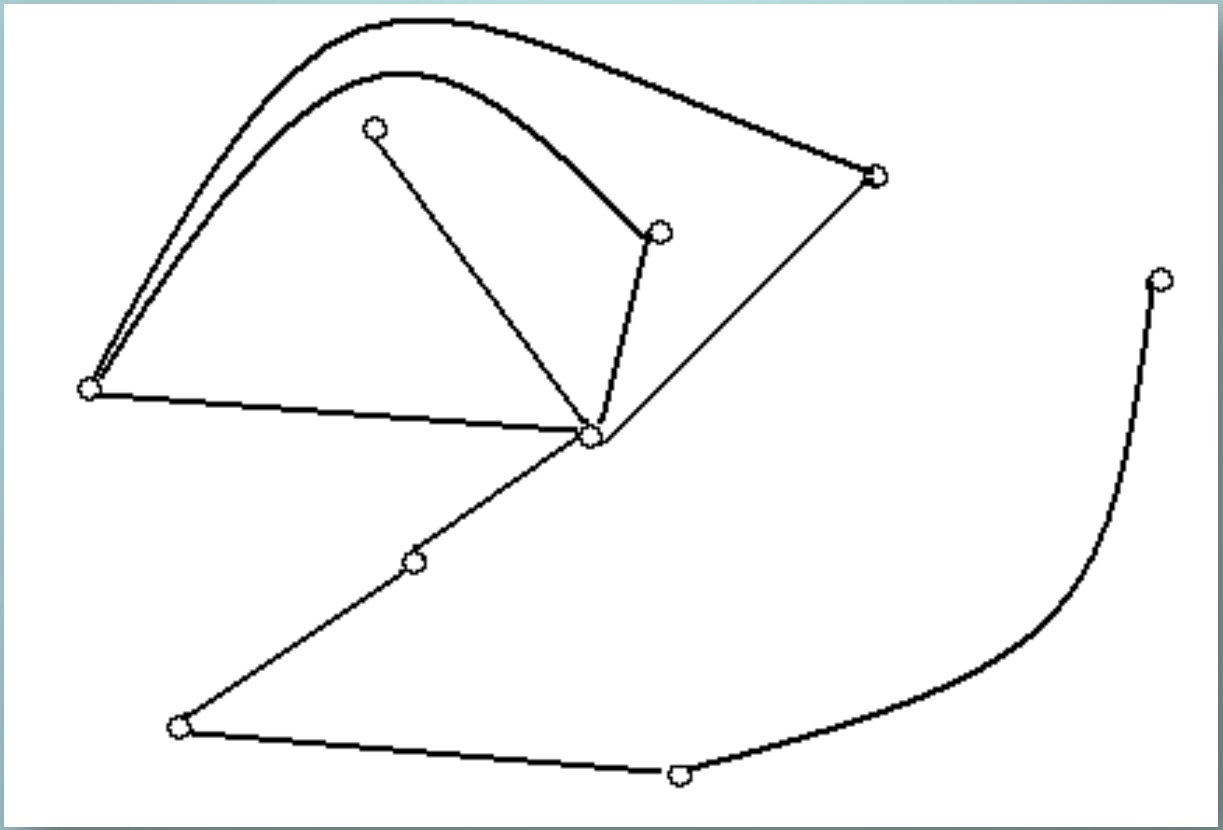
Proof by induction on the number of vertices.

- **Base Case:** Any connected simple planar graph with 5 or fewer vertices is 5-colorable.
- **Induction Hypothesis:** Assume every connected simple planar graphs with k vertices is 5-colorable.
- Prove for a graph with $k+1$ vertices.

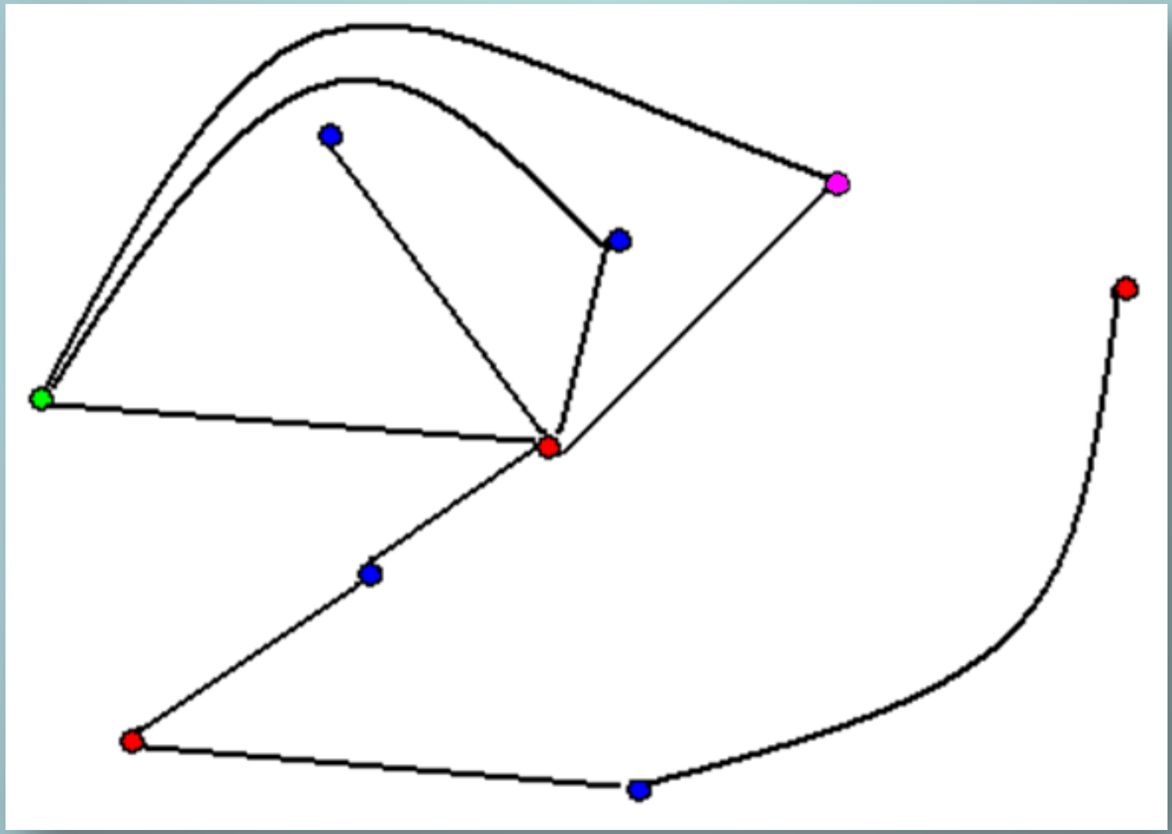
Let G be an SCP graph with (k+1) vertices.   It contains at least one vertex, V^*, with degree 5 or less.



- Remove this vertex and all edges incident to it.
- By the induction hypothesis the remaining graph with *k* vertices is 5-colorable.

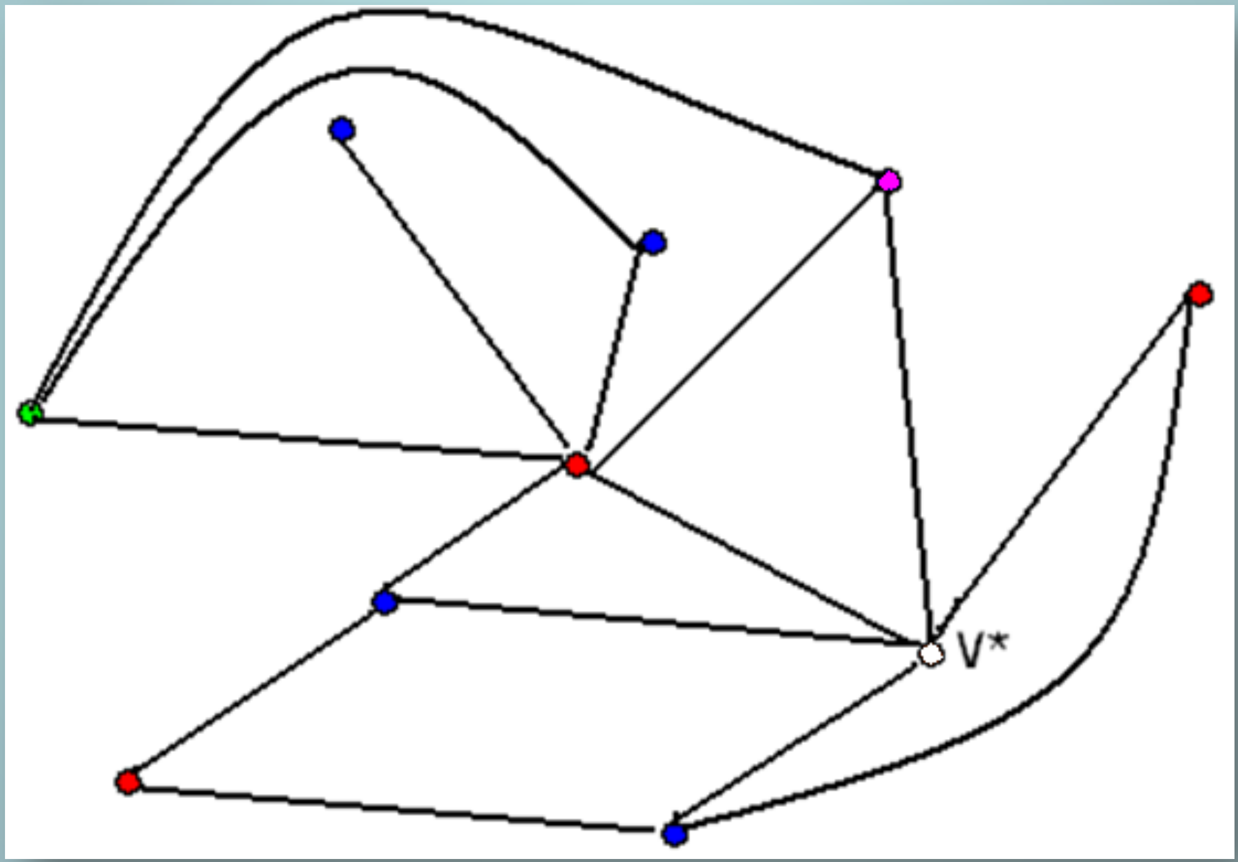


Color this graph with 5 colors.

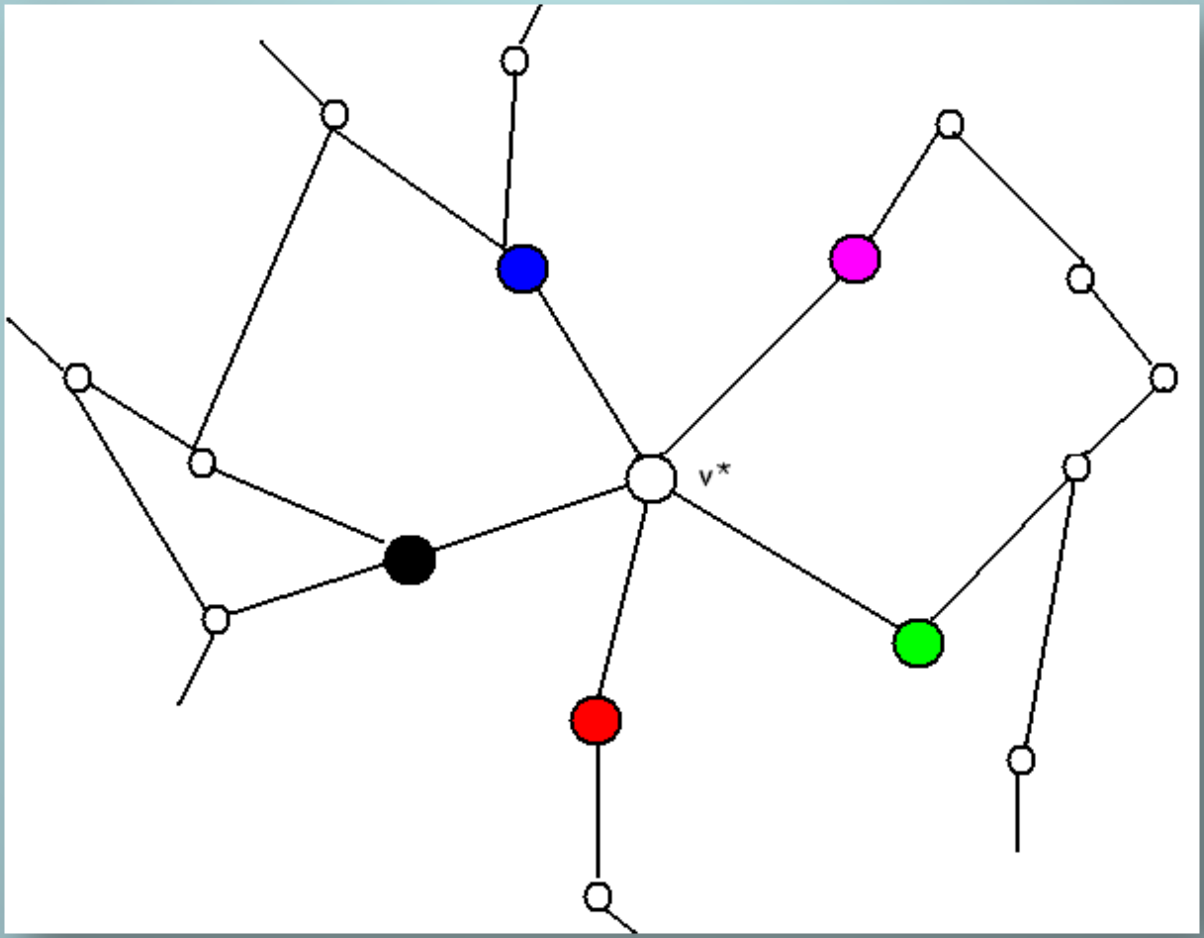


Replace *V** and the incident edges. We can color *V** only if its adjacent vertices do not use all 5 colors.

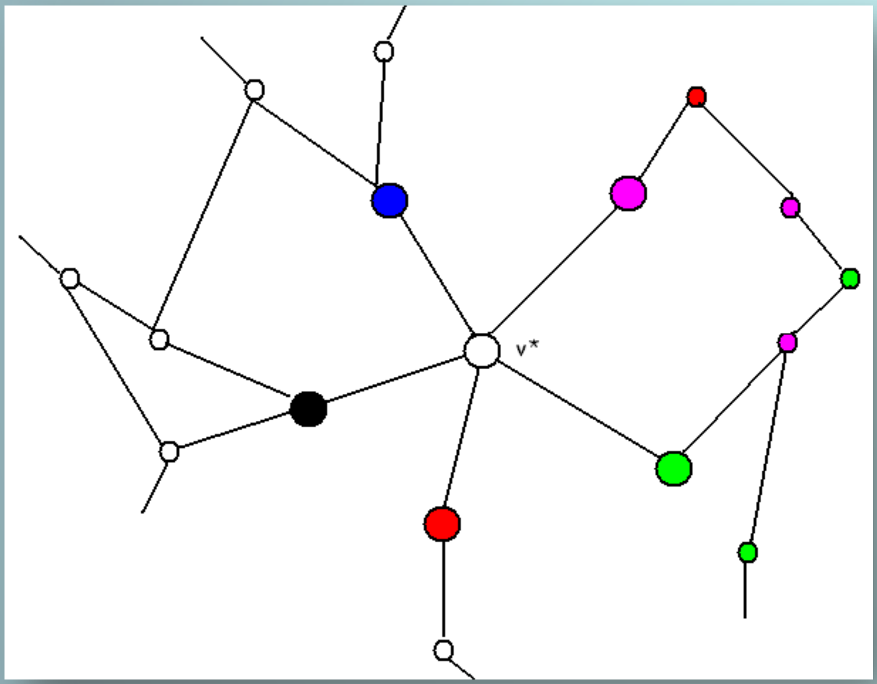
Therefore assume all 5 colors are already used.

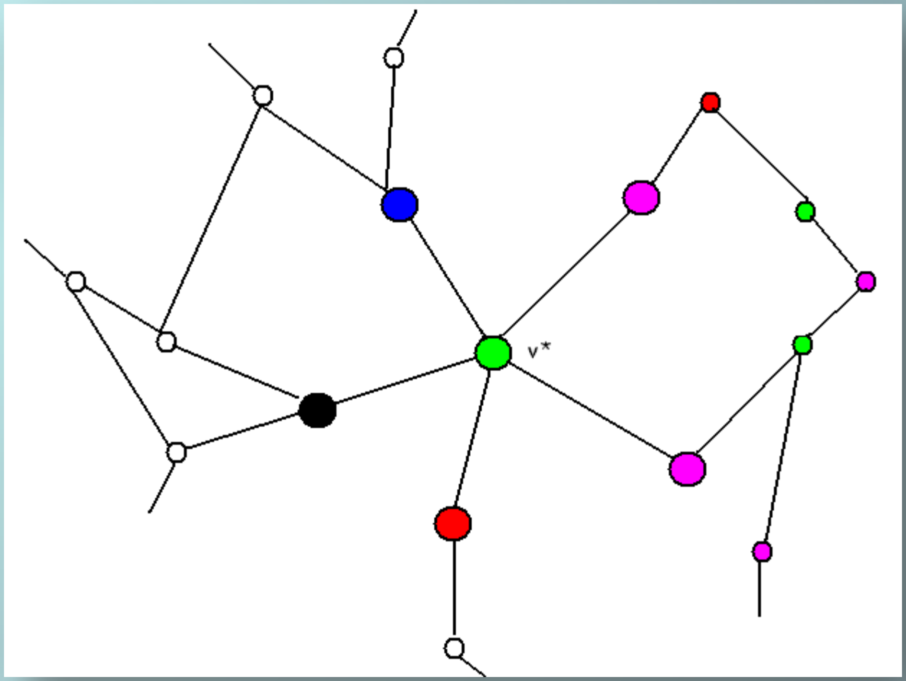


Consider all G-M paths leading out of V^*
(paths that alternate Green-Magenta-Green-Magenta...)

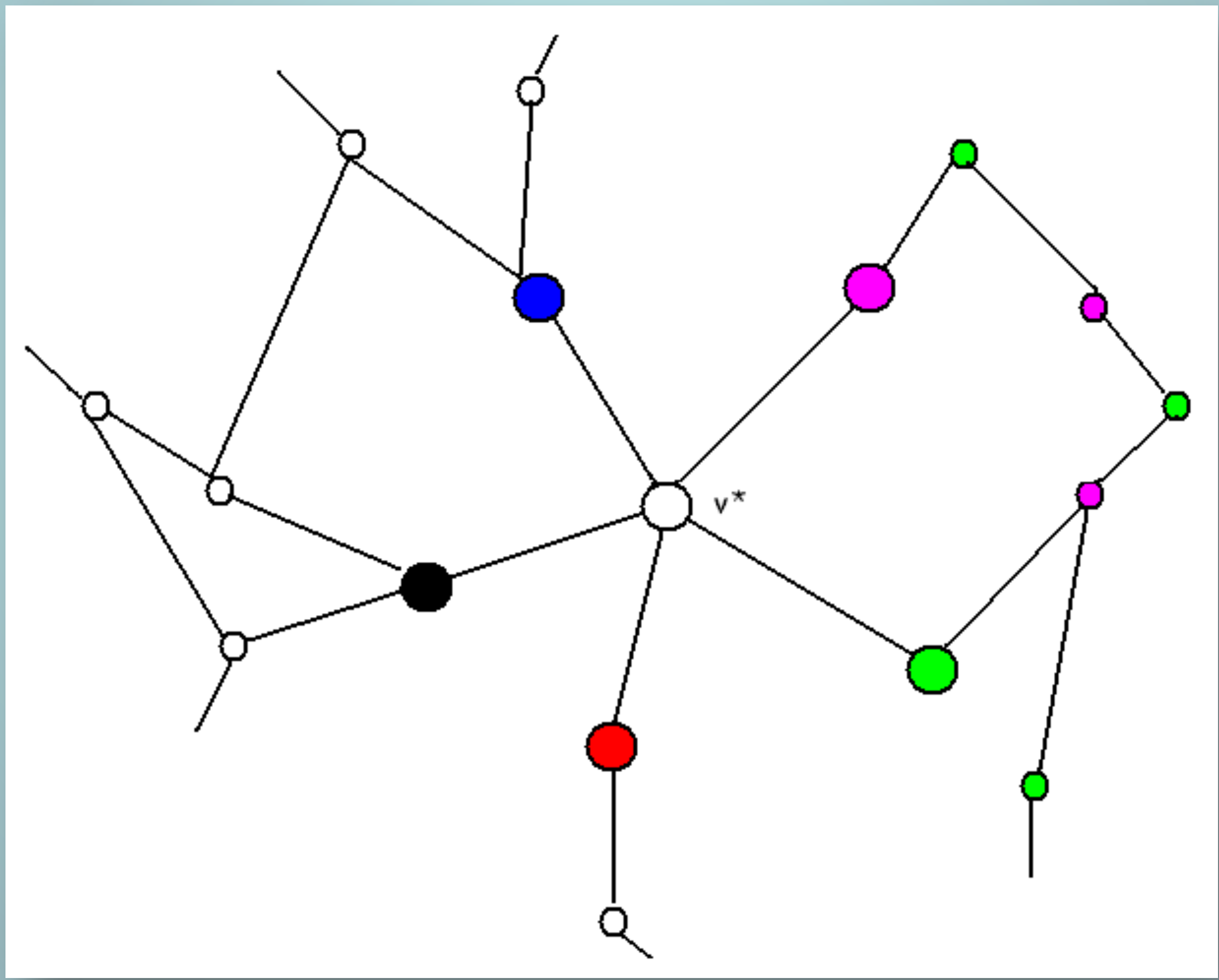


If there is no connected path leading back to V^* then switch M and G, and color V^* Green.

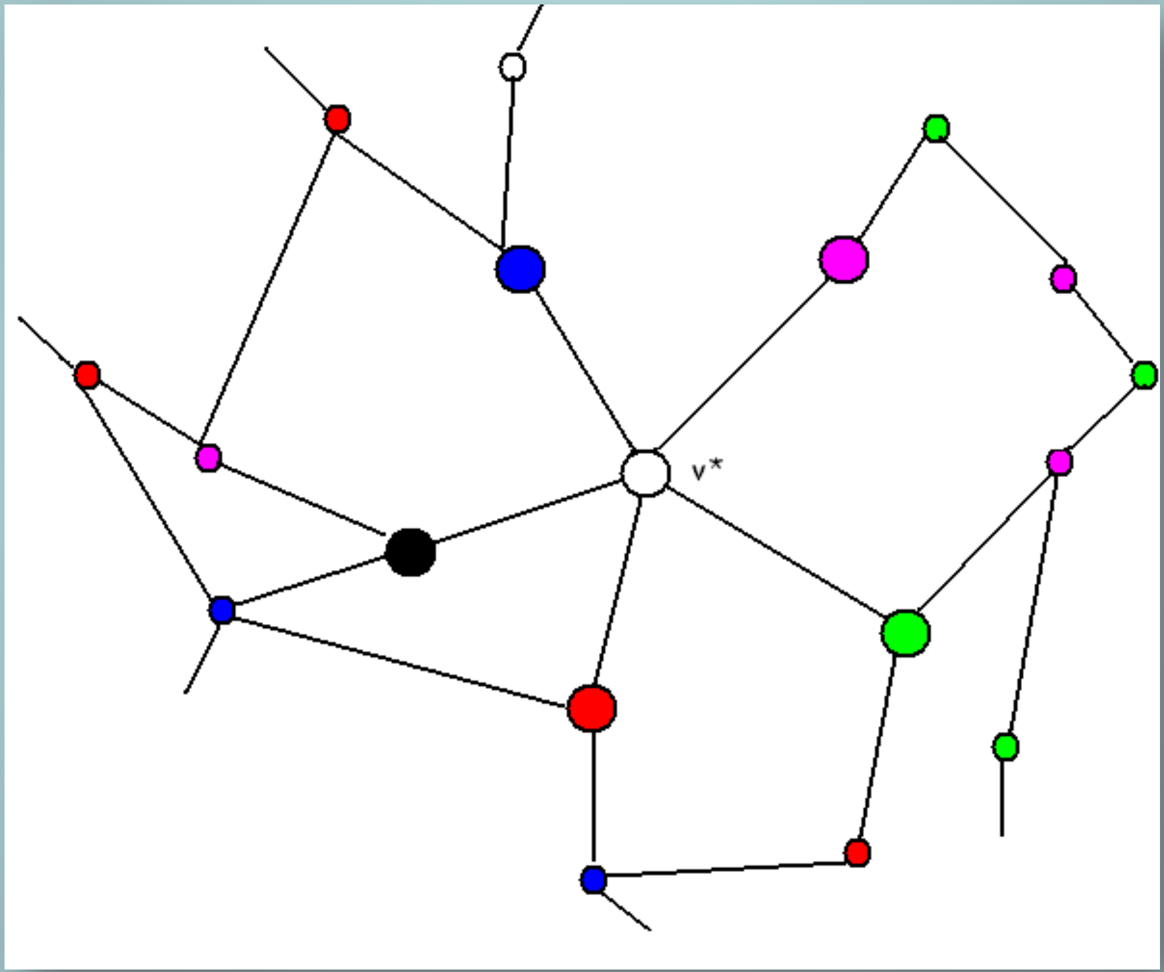




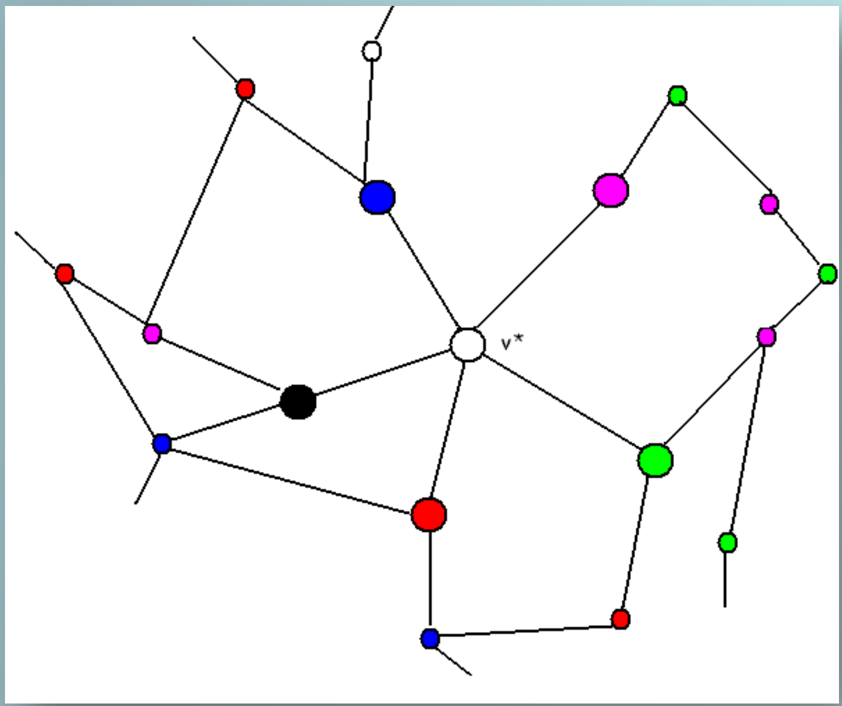
If there exists a connected path, then switching does not help.

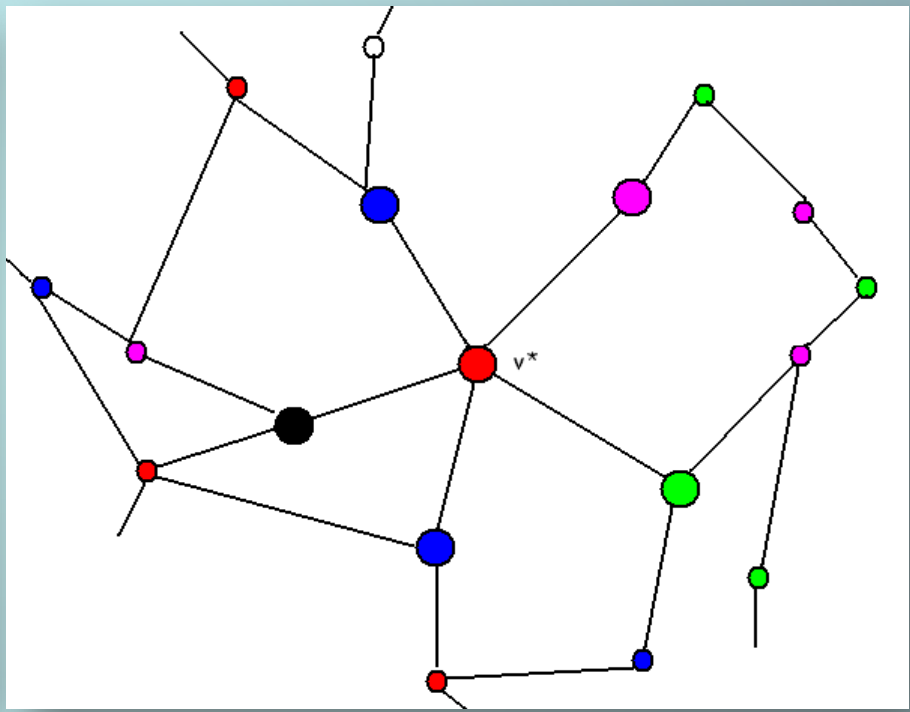


Is there a Red-Blue connecting path?

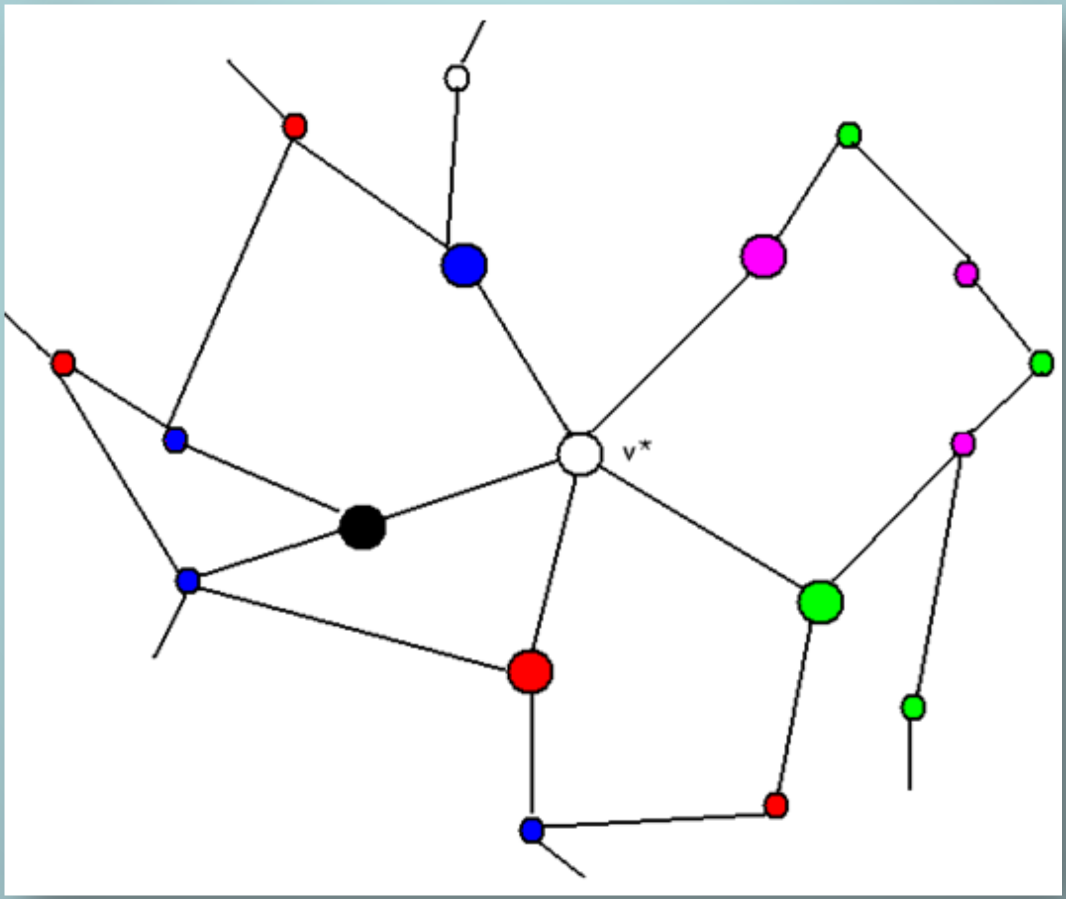


If not, switch the Red and Blue colored vertices and color *V** Red.

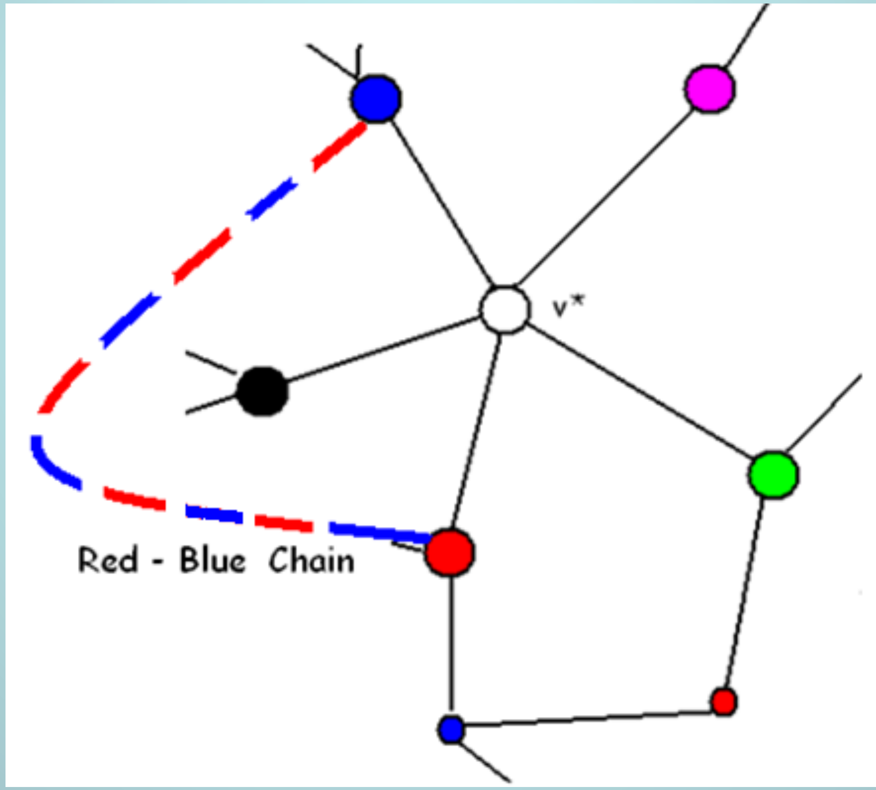




Therefore assume that there exists a Red-Blue connecting path.



If there is a Red-Blue Chain, there cannot be a Black – Green Chain, since it is blocked by the Red-Blue Chain.



Then switch the colors of the Black – Green chain and color *V** Black