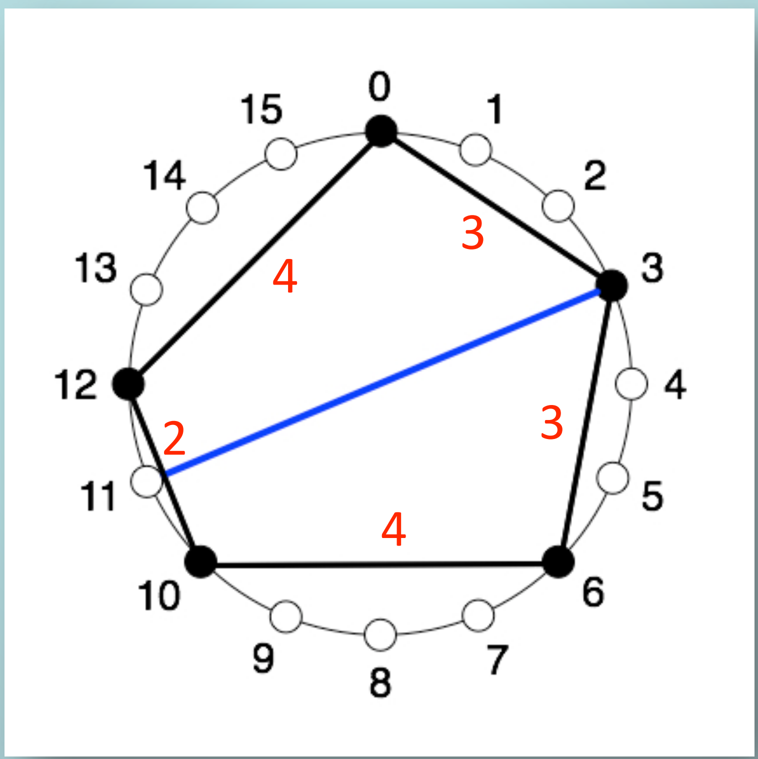
Multisets: Applications to Music

Godfried Toussaint

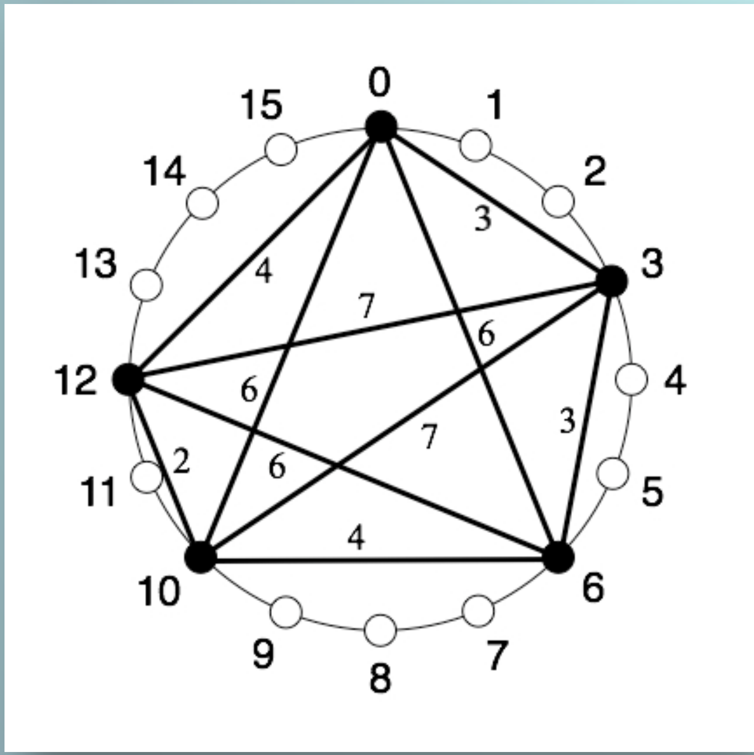
The clave son in *convex polygon* notation

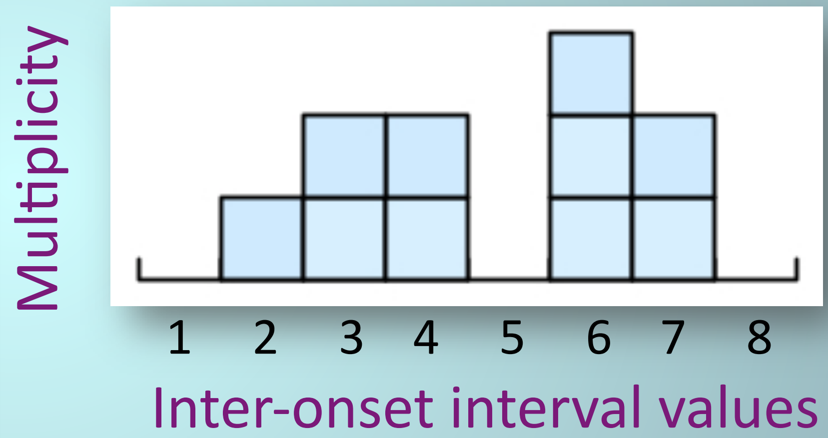
Easy to discover symmetries.



The clave son as a multiset: *inter-onset-interval histogram* notation

All the inter-onset intervals determine a multiset.





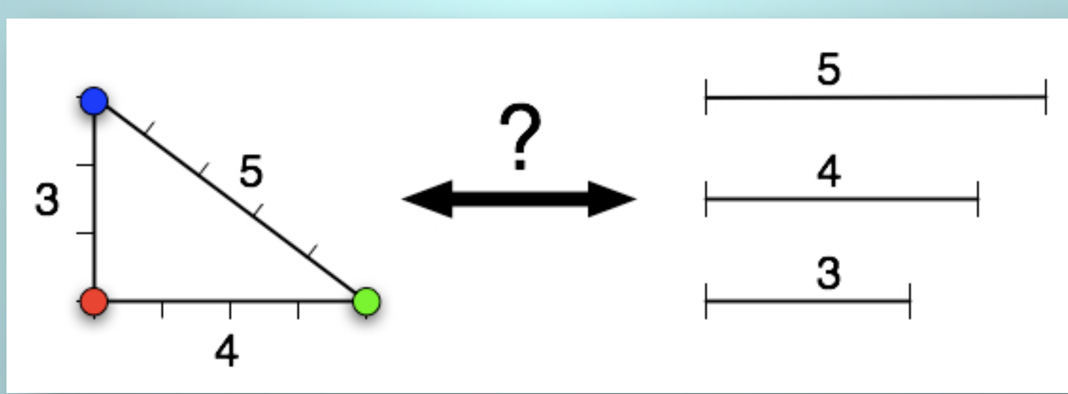
Question:

Should we use this more complete information?

X-ray crystallography in the 1920's

Distance geometry problem: Given a multiset of distances between a set of objects (without labels), can one reconstruct the spatial configuration of the objects, and if so, is the solution unique?

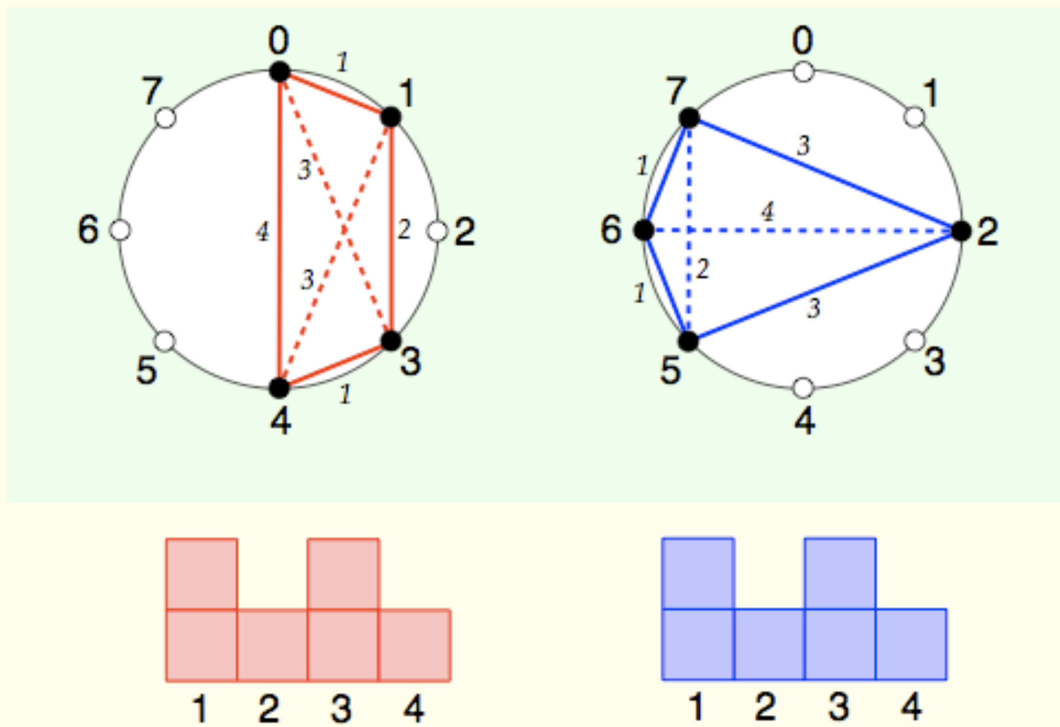
fictional 3-atom molecule



Distances between atoms

A. Lindo Patterson, "Ambiguities in the X-ray analysis of crystal structures," *Physical Review*, March, 1944.

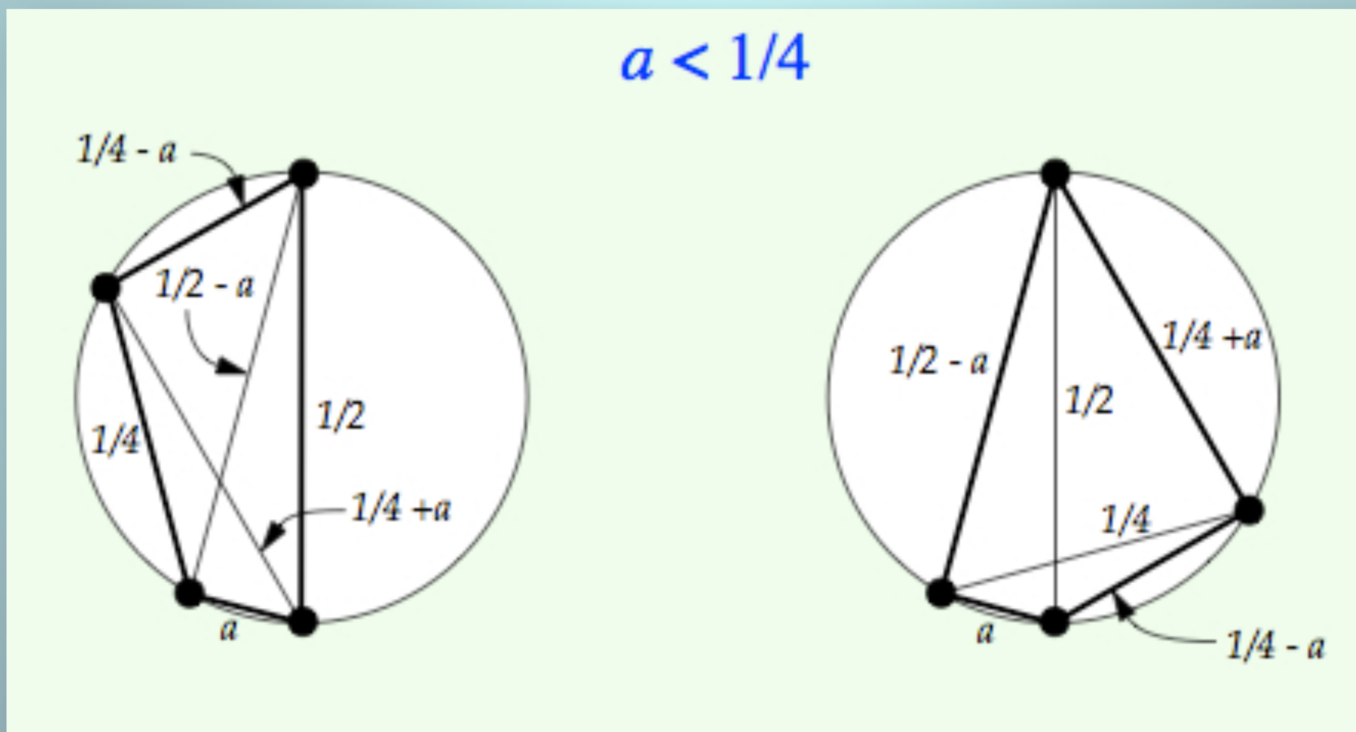
Different *cyclotomic* sets can have the same set of distances (homometric sets).



Interval histograms

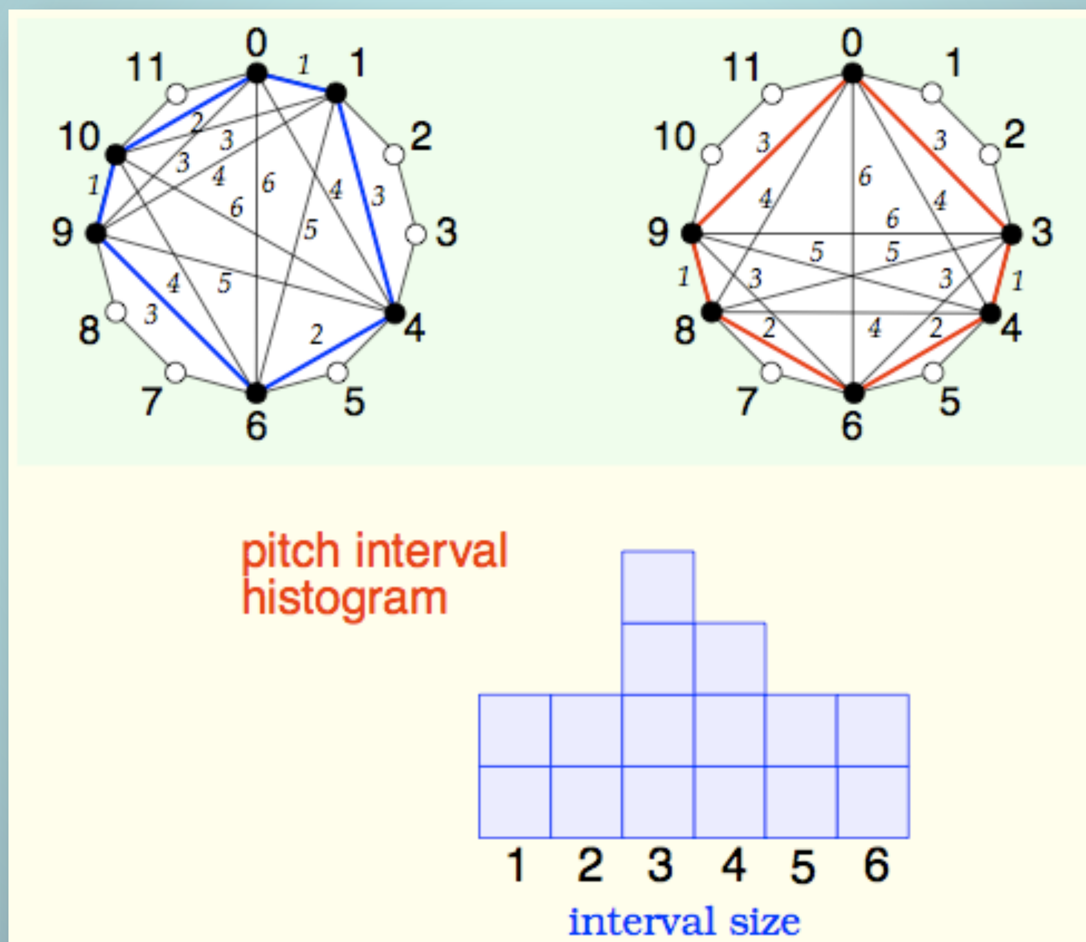
The infinite family of homometric pairs of Paul Erdős

Paul Erdős, in personal communication to
A. Lindo Patterson, *Physical Review,* March, 1944.



The Hexachordal Theorem

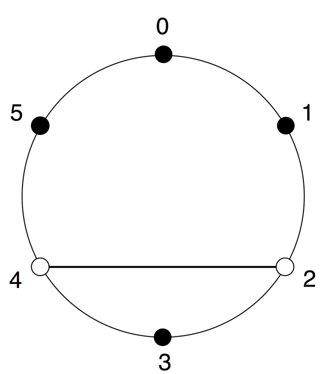
Two complementary hexachords have the same interval content.
First observed empirically: Arnold Schoenberg. ~ 1908



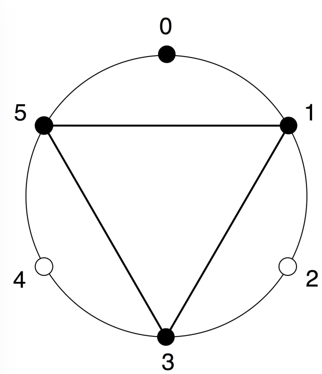
Induction Proof of the Hexachordal Theorem

Juan E. Iglesias, "On Patterson's cyclotomic sets and how to count them," *Zeitschrift für Kristallographie*, 1981.

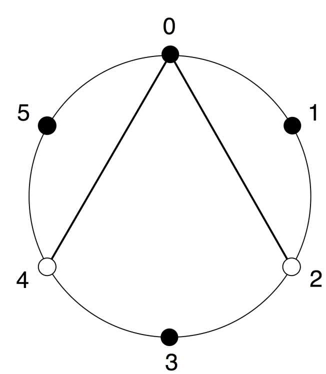
THEOREM: Let p of the N vertices of a regular polygon inscribed on a circle be black dots, and the remaining $q = N - p$ vertices be white dots. Let n_{ww}, n_{bb}, and n_{bw} denote the multiplicities of the distances of a *specified length d* between **white-white**, **black-black**, and **black-white**, vertices, respectively. Then the following relations hold:



for d = 2, N(ww) = 1



for d = 2, N(bb) = 3



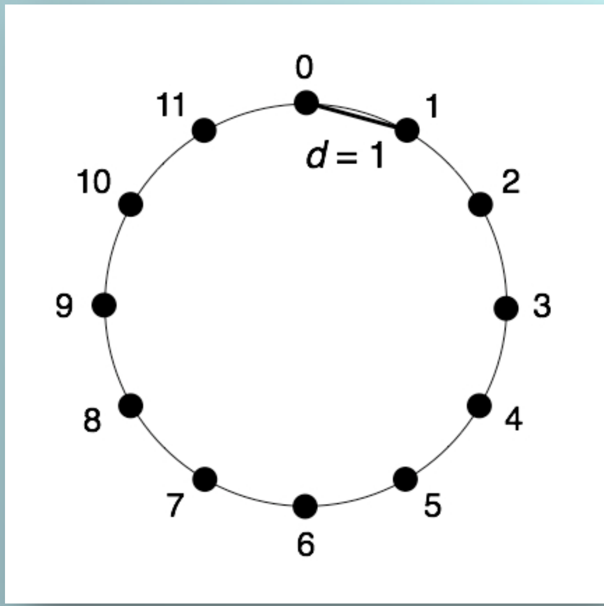
for d = 2, N(bw) = 2

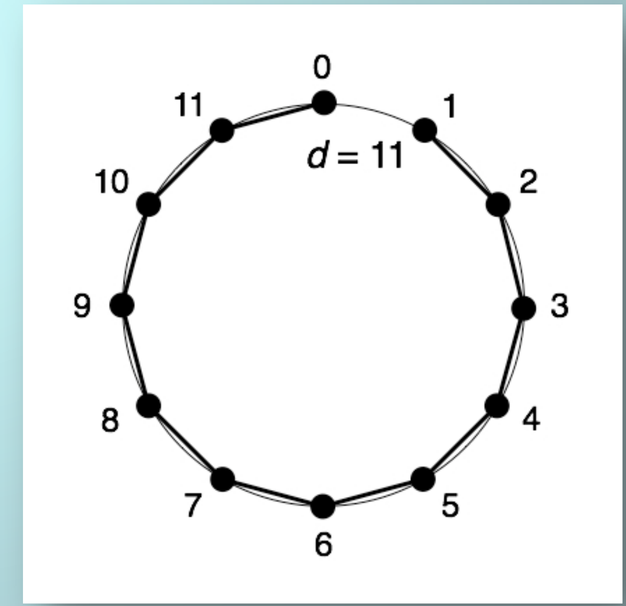
$$p = n_{bb} + (1/2)n_{bw}$$
$$q = n_{ww} + (1/2)n_{bw}$$

Induction Proof of the Hexachordal Theorem:
Base case when all the vertices are black.

LEMMA: Each distance value d occurs n times.

If d = 1 or d = n-1, its multiplicity equals the number of of sides of an n-vertex regular polygon.





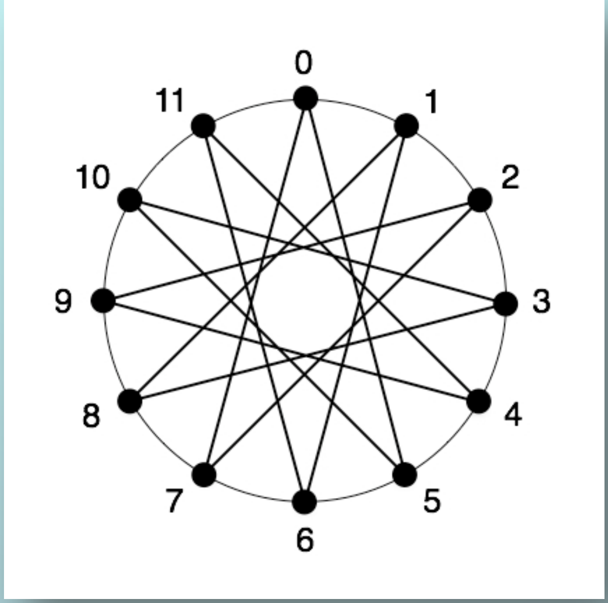
Example with n = 12

Induction Proof of the Hexachordal Theorem:
Base case when all the vertices are black.

Lemma: Each distance value *d* occurs *n* times.

If 1 < *d* < *n*-1, and *d* and *n* are relatively prime, the multiplicity of *d* equals the number of of sides of an *n*-vertex regular star polygon.

Example: *n* =12
d = 5

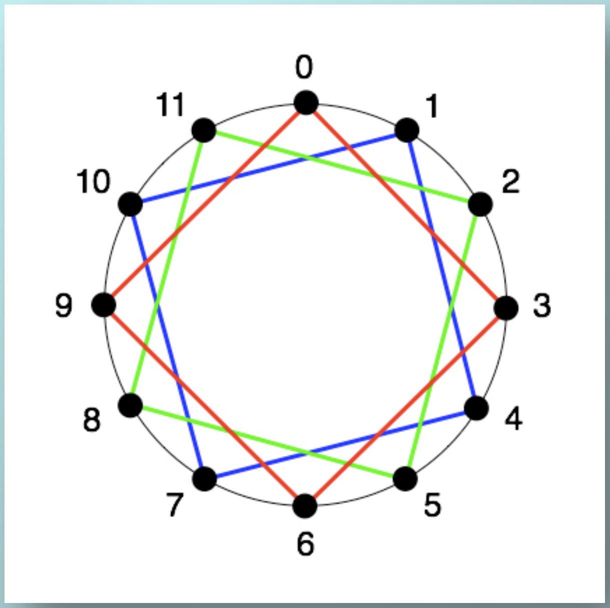


Induction Proof of the Hexachordal Theorem:
Base case when all the vertices are black.

Lemma: Each distance value *d* occurs *n* times.

If *d* and *n* are not relatively prime then the multiplicity of *d* equals the total number of sides of a group of regular polygons. There are g.c.d.(*d*,*n*) polygons with *n*/g.c.d(*d*,*n*) sides each.

Example: *n* =12
d = 3

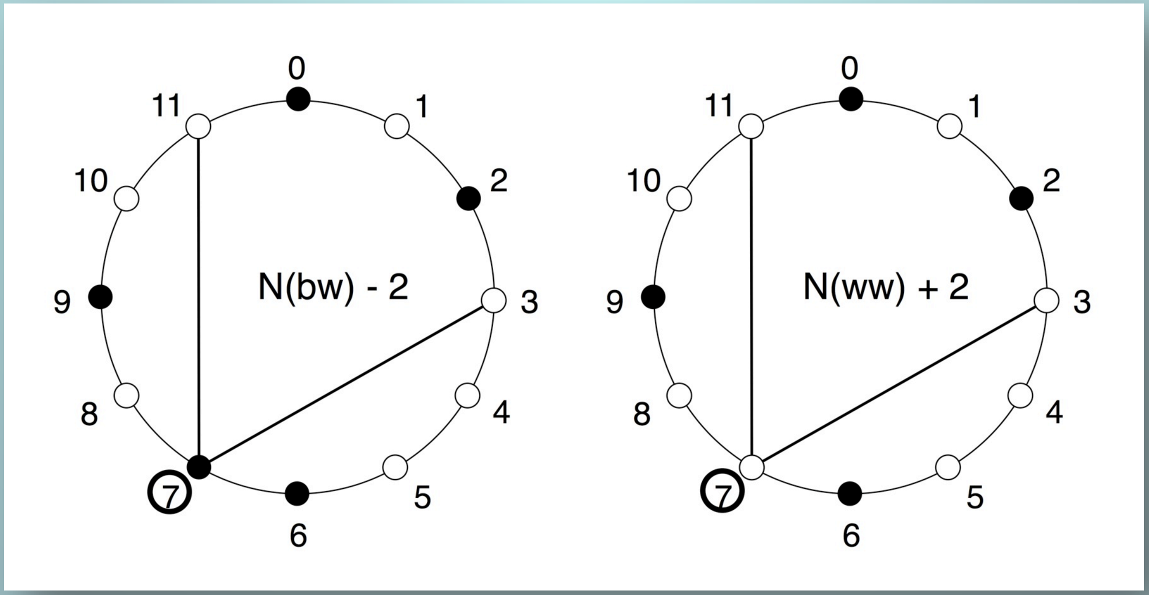


Induction Proof of the Hexachordal Theorem: General step for each value of *d*.

$$p = n_{bb} + (1/2)n_{bw}$$
$$q = n_{ww} + (1/2)n_{bw}$$

Case 1:

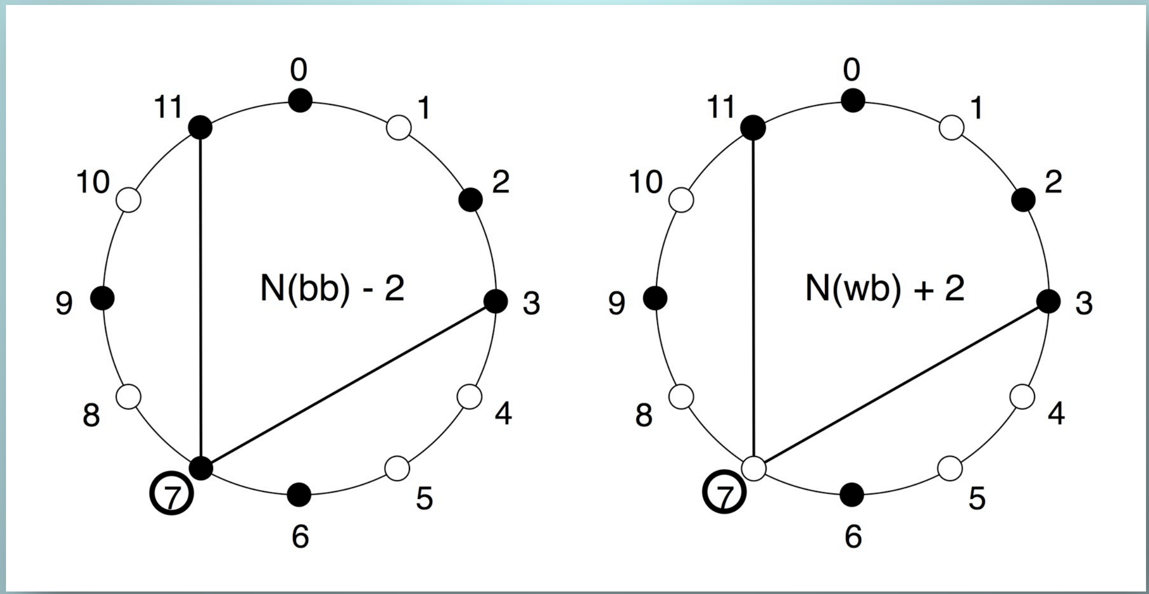
Both neighbors at distance *d* are white.



Induction Proof of the Hexachordal Theorem:
General step for each value of *d*.

$$p = n_{bb} + (1/2)n_{bw}$$
$$q = n_{ww} + (1/2)n_{bw}$$

Case 2: Both neighbors at distance *d* are black.

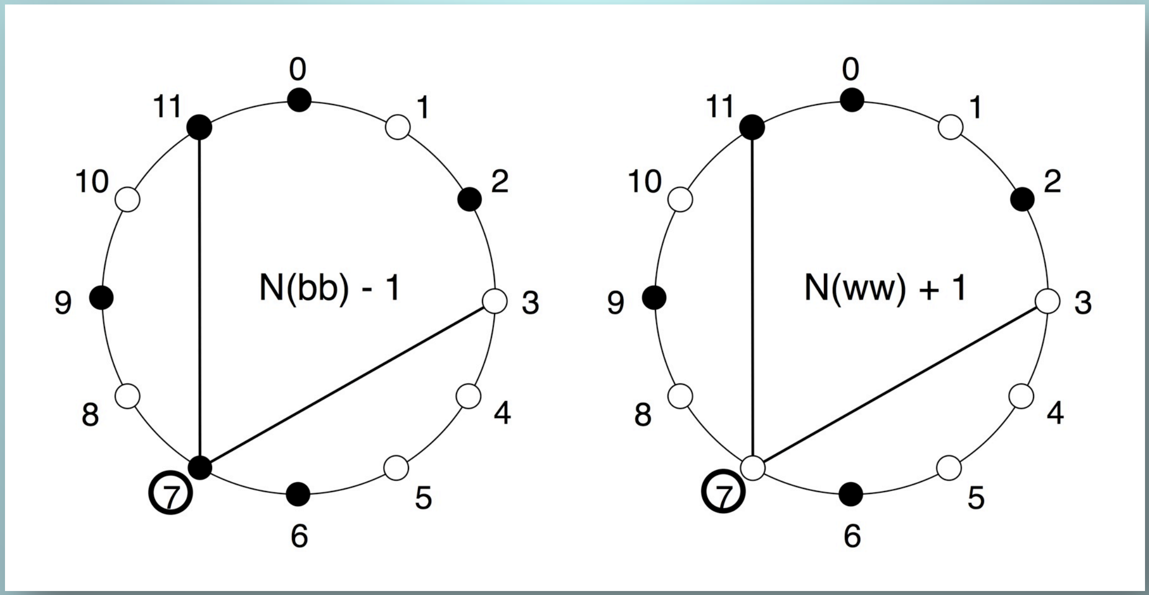


Induction Proof of the Hexachordal Theorem:
General step for each value of *d*.

$$p = n_{bb} + (1/2)n_{bw}$$
$$q = n_{ww} + (1/2)n_{bw}$$

Case 3:

One neighbor at distance *d* is black and the other white.



Induction Proof of the Hexachordal Theorem:
Corollary Step

Corollary: If $p = q$ then the two sets are homometric.

Proof:

$$p = n_{bb} + (1/2)n_{bw}$$
$$q = n_{ww} + (1/2)n_{bw}$$

if $p = q$ then

$$n_{bb} + (1/2)n_{bw} = n_{ww} + (1/2)n_{bw}$$

and

$$n_{bb} = n_{ww}$$

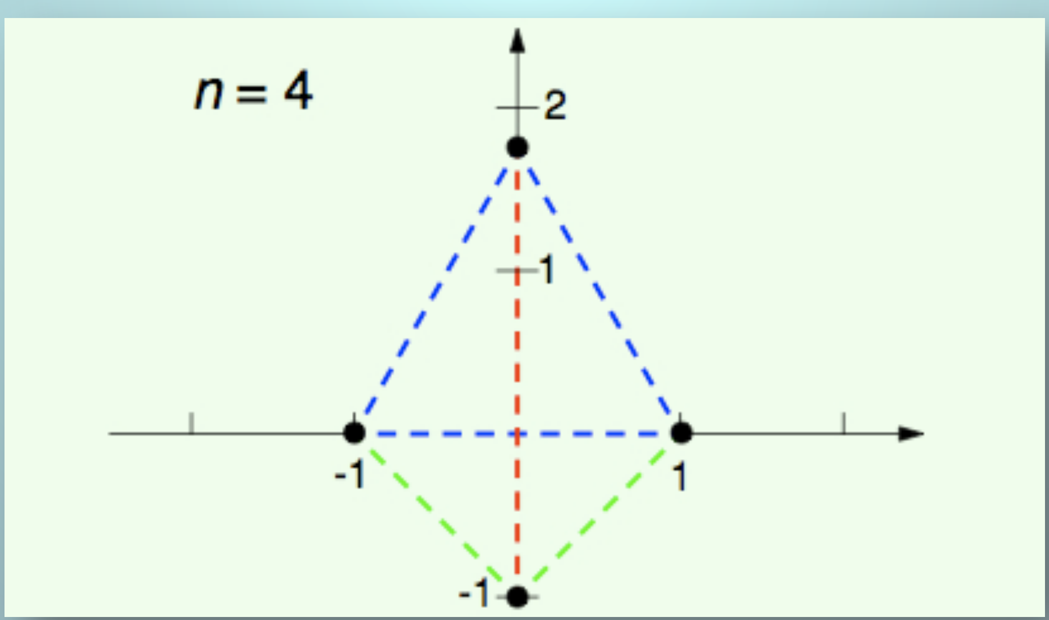
Points with Specified Distance Multiplicities

Paul Erdős -1986

Can one find n points in the plane (no 3 on a line and no 4 on a circle) so that for every $i = 1, 2, ..., n-1$ there is a distance determined by these points that occurs exactly i times?

Solutions have been found for $n = 2, 3, \dots, 8$.

Ilona Palásti for $n = 7$ and 8.



Winograd-Deep Scales

Deep scales have been studied in music theory since at least 1967.
Terry Winograd and Carlton Gamer, *Journal of Music Theory*, 1967.

Every inter-pulse interval in the circular lattice is realized by a pair of onsets, and it occurs a *unique number of times*.