Proof of Ore's Theorem by Backwards Induction

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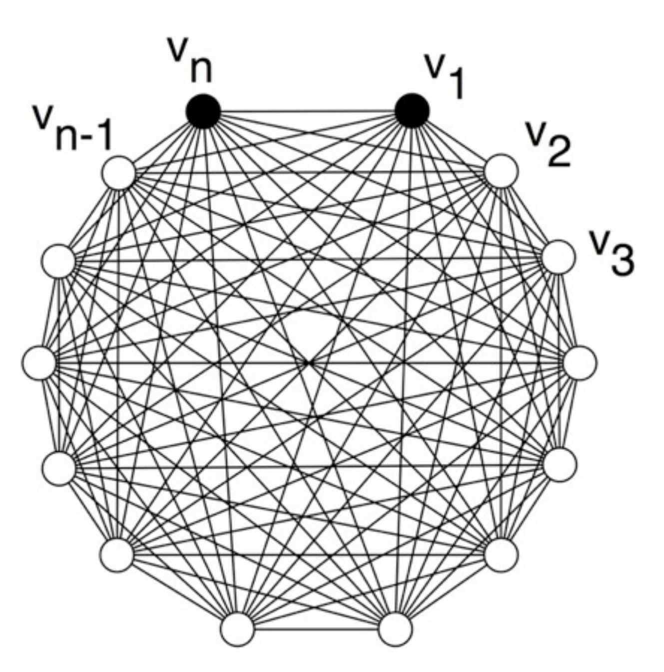
Ore's Theorem – Combining Backwards Induction with the Pigeonhole Principle

Let $G = (V, E)$ be a connected simple graph with $n \geq 3$ vertices. If G has the property that for each pair of non-adjacent vertices $u, v \in V$, we have that **deg u + deg $v \geq$ n** then G contains a Hamiltonian cycle.

Proof: by backwards induction on the number of edges in E:

Base case: G_C is the complete graph with n(n-1)/2 edges.



Connect the vertices in G_C in any order such as $(v_1, v_2, \ldots, v_n)$ to create a Hamiltonian path, and add edge (v_n, v_1) to create a Hamiltonian cycle.

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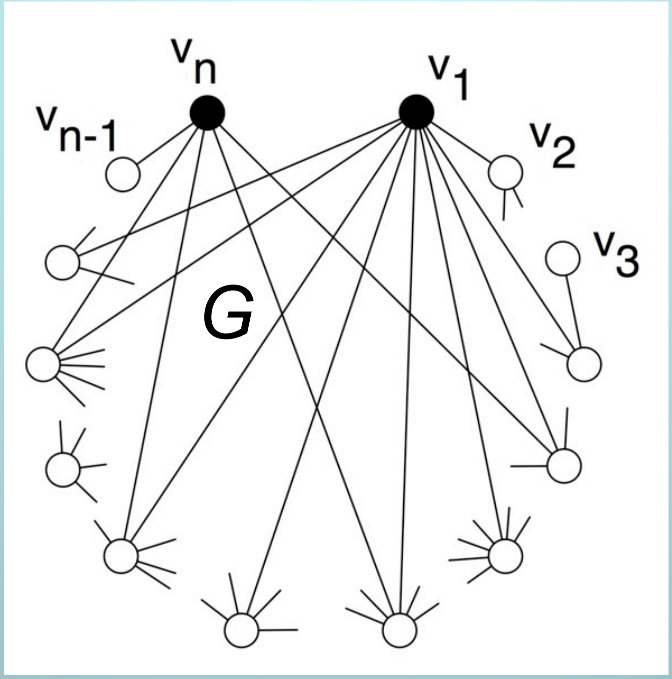
Induction hypothesis: the theorem is true when *G* has k edges.

- We must prove the theorem when *G* has k-1 edges.
- Let *G* be such a graph, and let v_n and v_1 be a pair of non-adjacent vertices in *G* such that $\mathbf{deg\ v_n + deg\ v_1 \geq n}$.



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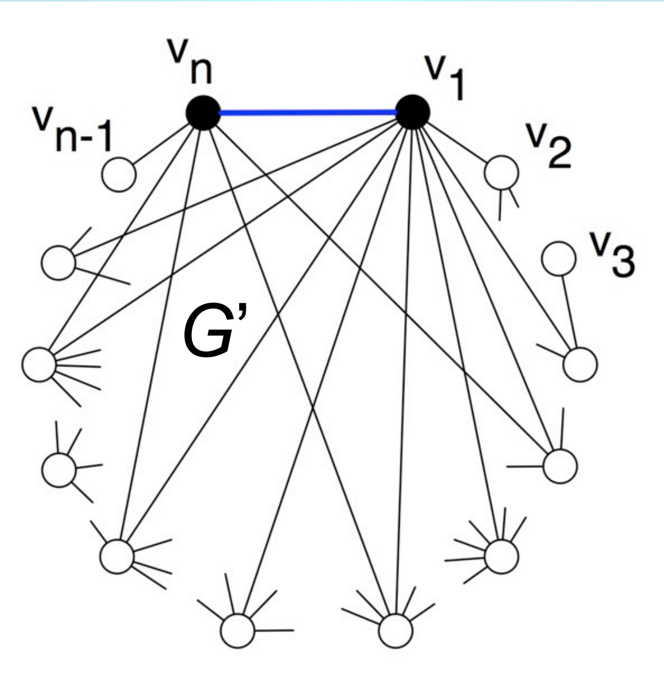
Induction hypothesis: the theorem is true when *G* has k edges.

- Let G' be the graph obtained by adding an edge between v_n and v_1 in *G. G'* therefore has k edges.
- It follows from the induction hypothesis that G' contains a Hamiltonian cycle.



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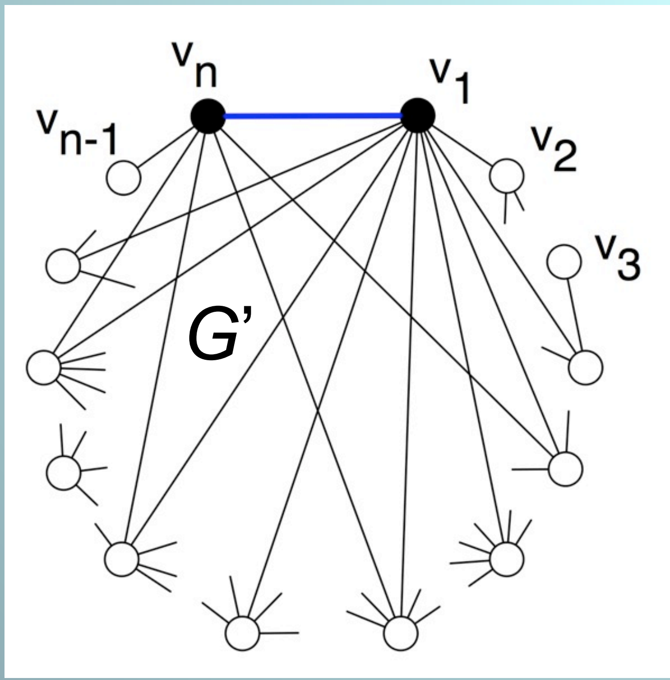
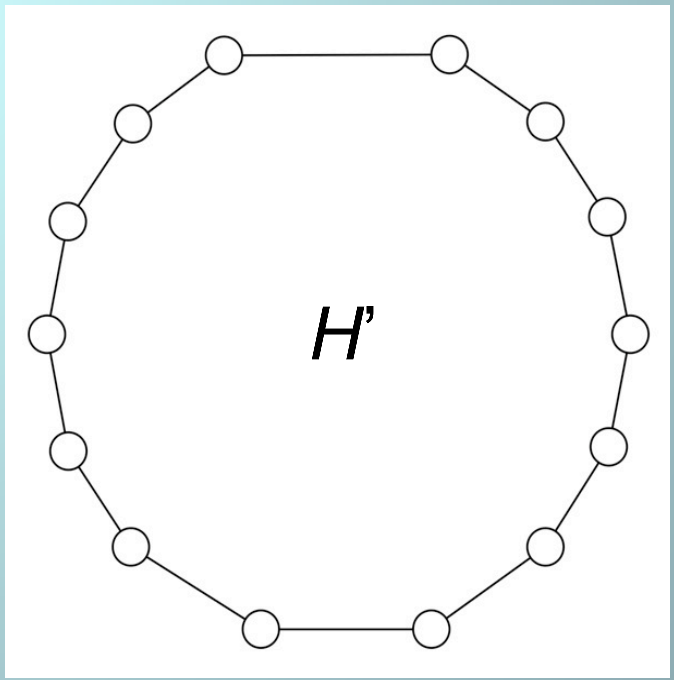
Let H' be the Hamiltonian cycle in *G'*.

We must now remove the edge (v_n, v_1) from *G'* to restore *G*.
Two cases arise:
Case 1: *H'* does not contain (v_n, v_1) . Then *H'* is a Hamiltonian cycle in *G*, and we are done. Edge (v_n, v_1) may be safely removed from *G'*.





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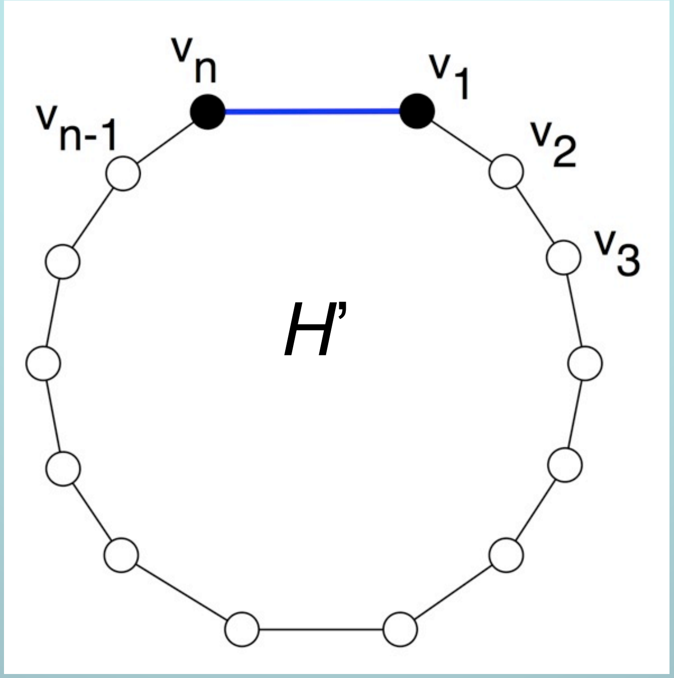
Case 2: *H'* contains (v_n, v_1).

•Without loss of generality let H' = (v_1, v_2, … , v_n, v_1).

•Delete the edge (v_n, v_1) from *G'* to recover *G*.



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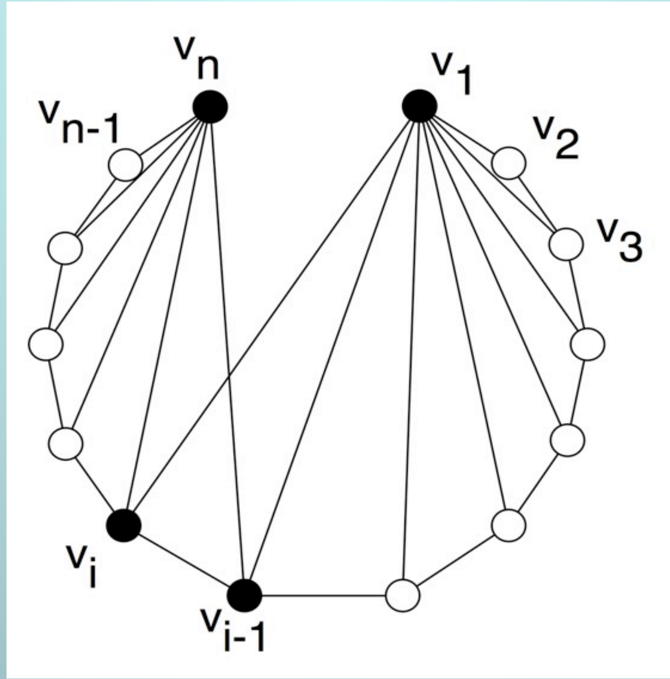
Case 2: *continued…*

Since **deg v_n + deg $v_1 \geq$ n** it follows from the Pigeonhole Principle that here must exist vertices v_{i-1} and v_i such that v_{i-1} is connected to v_n and v_i is connected to v_1 .



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Case 2: *continued…*

Therefore G contains the Hamiltonian cycle
$H = (v_1, v_2, \ldots, v_{i-1}, v_n, v_{n-1}, v_{n-2}, \ldots, v_i, v_1)$. Q.E.D.