Interlocking Rhythms, Duration Interval Content, Cyclotomic Sets, and the Hexachordal Theorem

Godfried T. Toussaint

School of Computer Science, McGill University, 3480 University St., Montreal, Canada
[godfried@cs.mcgill.ca]

2000 Mathematics Subject Classification. 68R01, 52C99, 52C10, 90B80, 51K99

We consider a rhythm to be represented by a subset of k points (onsets) on the circular lattice consisting of n points equally spaced on a circle [1]. Such sets are called *cyclotomic* sets in the crystallography literature [3], [13]. Every pair of such points determines an inter-onset-duration-interval (the geodesic distance between the pair of points on the circle) [14]. The histogram of this multiset of distances in the context of musical scales and chords is called its *interval content* [2]. We will use the same terminology for the case of rhythms, where the intervals are durations of time. The remaining $(n-k)$ lattice points determine a complementary rhythm [6]. A rhythm and its complement may thus be viewed as a pair of *interlocking* rhythms. Two rhythms which are not congruent but possess the same multiset of distances are said to be *homometric*, a term introduced by Lindo Patterson in 1939 [4].

First we review some of the history of interlocking rhythms in African and European music. Then we review the history of the hexachordal theorem in music theory [8], [9], [10], [11]. The hexachordal theorem states that two non-congruent interlocking rhythms with $k = n/2$ are homometric. The earliest proof of this theorem in the music literature appears to be due to Milton Babbitt and David Lewin [15], [8], [9], [10]. It used heavy machinery from topology. Later Lewin obtained new proofs using group theory. Later still Eric Regener [5] found an elementary simple proof of this theorem. The music theorists appear to be unaware that this theorem was known to crystallographers at least twenty years earlier [3]. It seems to have been proved by Lindo Patterson [3] around 1940 but he did not publish a proof. In the crystallography literature the theorem is called Patterson's theorem [12]. The first published proof in the crystallography literature is due to Buerger [12]; it is based on image algebra, and is non-intuitive. A much simpler and elegant elementary proof was later found by Iglesias [13]. The simplest elementary proof was published by Steven Blau in 1999 [7]. We elucidate the proofs by Iglesias [13] and Blau [7].

[1] Toussaint, Godfried T., The geometry of musical rhythm. In *Proceedings of the Japan Conference on Discrete and Computational Geometry*, (eds. Jin Akiyama, Mikio Kano, and Xuehou Tan). LNCS 3742, Springer-Verlag, Berlin-Heidelberg, 2005, 198–212.

[2] Lewin, David, On the interval content of invertible hexachords. *Journal of Music Theory*, Vol. 20, No. 2, Autumn, 1976, pp. 185–188.

[3] Patterson, A. Lindo, Ambiguities in the X-ray analysis of crystal structures. *Physical Review*, Vol. 64, Nos. 5-6, March 1944, pp. 195–201.

[4] Mardix, S., Homometrism in close-packed structures. *Acta Crystallographica*, Vol. A46, 1990, pp. 133-138.

[5] Regener, Eric, On Allen Forte's theory of chords. *Perspectives of New Music*, Vol. 13, No. 1, Autumn-Winter 1974, pp. 191–212.

[6] Morris, Robert D., Pitch-class complementation and its generalizations. *Journal of Music Theory*, Vol. 34, No. 2, Autumn, 1990, pp. 175–245.

[7] Blau, Steven K., The hexachordal theorem: a mathematical look at interval relations in twelve-tone composition. *Mathematics Magazine*, Vol. 72, No. 4, October 1999, pp. 310–313.

[8] Lewin, David, Intervallic relations between two collections of notes. *Journal of Music Theory*, Vol. 3, No. 2, November, 1959, pp. 298–301.

[9] Lewin, David, The intervallic content of a collection of notes, intervallic relations between a collection of notes and its complement: an application to Schoenberg's hexachordal pieces. *Journal of Music Theory*, Vol. 4, No. 1, April, 1960, pp. 98–101.

[10] Lewin, David, *Generalized Musical Intervals and Transformations*. Yale University Press, New Haven.

[11] Soderberg, Stephen, Z-related sets as dual inversions. *Journal of Music Theory*, Vol. 39, No. 1, Spring, 1995, pp. 77–100.

[12] Buerger, Martin J., Proofs and generalizations of Patterson's theorems on homometric complementary sets. *Zeitschrift für Kristallographie*, Vol. 143, 1976, pp. 79–98.

[13] Iglesias, Juan E., On Patterson's cyclotomic sets and how to count them. *Zeitschrift für Kristallographie*, Vol. 156, 1981, pp. 187–196.

[14] Buerger, Martin J., Interpoint distances in cyclotomic sets. *The Canadian Mineralogist*, Vol. 16, 1978, pp. 301–314.

[15] Milton, Babbitt, in *Words About Music*. Eds. S. Dembski and J. N. Straus. University of Wisconsin Press, Madison, WI, 1986, p. 105.