Nice Perspective Projections*

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A polyhedral object in three-dimensional space is often well represented by a set of points and line segments that act as its features. By a nice perspective projection of an object we mean a projection that gives an image in which the features of the object, relevant for some task, are visible without ambiguity. In this paper we consider the problem of computing a variety of nice perspective projections of three-dimensional objects such as simple polygonal chains, wire-frame drawings of graphs and geometric rooted trees. These problems arise in areas such as Computer Vision, Computer Graphics, Graph Drawing, Knot Theory and Computational Geometry.

1. INTRODUCTION

When we draw or plot an image of a three-dimensional ($3D$) object on a sheet of paper, or when we use a displaying device, such as a computer graphics screen, we obtain a 2D representation that necessarily approximates the $3D$ object and will never capture all its properties. It is obviously desirable to make this single

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image, even incomplete, as faithful as possible, where the measure of faithfulness may change for different fields. In order to obtain a (geometric) image of a $3D$ geometric object, we use a model (pinhole camera) based on *perspective projections*. A perspective projection of a $3D$ object gives a planar description of how the object looks from a particular view point in space, and we are interested in making that projection as good as possible. The topic of this paper is hence related to the much broader domain of Scientific Visualization [1, 2], the discipline concerned with helping us to better visualize information, objects and processes in their fullest generality.

We consider objects which may be abstracted as sets of segments in space. Ambiguities and loss of information can arise in the image (the perspective projection), and one is interested in producing projections which are the nicest in some sense. Besides the general case of sets of points and segments in space, we consider also the specific case of $3D$ simple polygonal chains, $3D$ wire-frame drawings of graphs and $3D$ geometric rooted trees.

There are many possible definitions of the notion of *quality* or *niceness* of an image of a $3D$ object obtained by a projection. There exists a variety of specific geometrical characteristics that can more accurately describe the niceness of a projection of an object. Some of these criteria are more desirable than others depending on the application on mind. A common sense requirement of quality is that the significant features of the $3D$ object be visible in the image. For example, it is natural to ask for the projection of a segment not be reduced to a point. It is also a normal requirement to try to minimize the number of crossings in the projection, making the image as simple and readable as possible. The term *nice projection* refers to these requirements and to many others, and was first introduced in [3, 4] where only orthogonal projections where considered.

Nice projections have received different names and definitions depending on the context where they are studied: regular projections in Knot Theory [5], Witinger projections in Scientific Visualization [6], general position projections in Computer Graphics [7], and non-degenerate projections in Computational Geometry [8, 9, 10]. Kamada and Kawai [7] give quality criteria related to the visualization of $3D$ objects represented by wire-frames and describe an algorithm for computing good directions for orthogonal projection. In [3, 11, 4] the authors consider orthogonal projections of sets of points, segments and polygonal objects; several criteria of niceness are introduced, and algorithms are described for optimizing those criteria. Eades, Houle and Webber in [12], and Webber in [13] extend and generalize the methods in [3, 4, 7] with the specific goal of computing good orthogonal projections in order to visualize $3D$ graph drawings.

While we see that extensive work has been done on the complexity of computing nice *orthogonal* projections, to date the more general problem of finding nice *perspective* projections has received meager attention in areas such as Computer Vision, Computer Graphics, Graph Drawing, Knot Theory and Computational Geometry. The quality of the images obtained by perspective projection depends strongly on the choice of the center of projection and sometimes, but not always, on the plane of projection. The situation is simpler for orthogonal projections, in particular because the choice of the viewpoint has one degree less of freedom, therefore it is not a surprise that the methods can not always be easily extended.

Computing perspective projections has many practical applications in Computer Vision problems that arise in robot navigation [22, 26, 28]. In this area most of the work has concentrated on defining models of degeneracy or types of "bad" projections [23, 24, 27].

In the above work the models were theoretical in the sense that points and lines have zero measure. In practice points and lines are made up of pixels and therefore do not have zero measure. This causes the probability of obtaining bad projections (or viewpoints) to increase. A practical and empirical study of this phenomenon was carried out in [25]. The computational complexity of determining nice *perspective* projections has not been addressed before.

In this paper we show how to obtain nice perspective projections under several measures of niceness: regularity (Section 3), simplicity and minimum-crossing (Section 4) and monotonicity (Section 5). Furthermore we study the computational complexity of obtaining such projections for the idealized case of inifinite resolution.

2. PERSPECTIVE PROJECTIONS

A perspective projection in space is fully determined by a point c, the *projection center*, and a plane π, the *projection plane*, that does not pass through c. Let π_c be the plane that goes through c and is parallel to π. The perspective projection of a point q, $q \notin \pi_c$, is the intersection q^* of the line qc with the plane π. The points of the plane π_c cannot be projected onto π.

The perspective projection of a geometric object is obtained by performing two operations: the computation of the lines that pass through the projection center c and each point of the object, and the intersection of this set of lines with the projection plane π.

3. REGULAR PROJECTIONS OF SEGMENTS

DEFINITION 3.1. Let S be a set of disjoint segments in space. A perspective projection of S is said to be **regular** when the following three conditions are fulfilled:

1.- no point of the projection plane is the projection of more than one endpoint of segments of S;

2.- no point of the projection plane is, simultaneously, the projection of an endpoint and an interior point of two segments of S;

3.- No point of the projection plane is the projection of more than two interior points of segments of S.

This definition implies that the projection of a segment should not be reduced to a point (condition 1), the endpoints of a segment should not be projected onto a point of another projected segment (conditions 1,2) and three projected segments should not intersect at an interior point (condition 3).

THEOREM 3.1. *Given a set of n disjoint segments in space, deciding whether their projection is regular can be done in $O(n^2)$ time and space.*

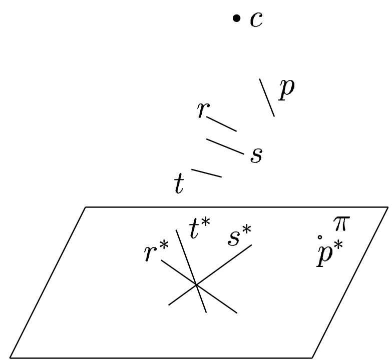

FIG. 1. *Non-regular segment projection*

Proof. Let S^* be the set of n segments that are the projection of the segments of S. It suffices to determine the forbidden situations using the technique of [16] that computes the arrangement of segments $\mathcal{A}(S^*)$ in $O(n^2)$ time and space. ■

A point that cannot be the projection center of a regular perspective projection of S is called a *forbidden point*. Let us to analyze the different situations in which forbidden points can arise. If a point lies on a line that passes through an endpoint of a segment and through a second point of the same or another segment, then the point will be a forbidden one. When the second point is an endpoint of the first or second segment, regularity condition 1 is not satisfied. If the second point is an interior point of the second segment, then regularity condition 2 is not satisfied. When a point lies on a line that simultaneously intersects three (or more) segments, then again the point is a forbidden one. Indeed, if we take that point as a projection center, then regularity condition 3 does not hold. These straight lines are called lines of forbidden points. Note that in all the cases the condition of regularity only depends on the position of the projection center in relation to the segments of S and it does not depend on the position of the projection plane.

In order to determine whether a line is a line of forbidden points, we must intersect the line with each of the n segments of S and then test if any of the three forbidden situations occurs. Therefore, we have the following:

OBSERVATION 3.1. *Given a set of n disjoint segments in space, deciding whether a line is a line of forbidden points can be done in $O(n)$ time and space.*

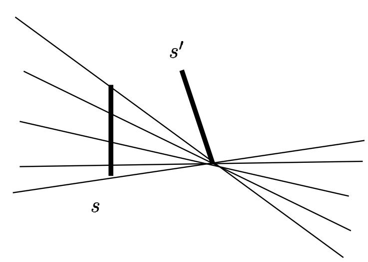

FIG. 2. *Double wedges*

Next we study the set of forbidden points in detail. The family of forbidden lines determined by one or two segments of S are their corresponding transversal lines. The transversal line of one segment is the line containing it. The transversal lines of two segments form four double wedges; each wedge is obtained by taking all the lines that go through an endpoint of a segment and through any point of the other segment (see Figure 2) . The set of forbidden lines given by three segments of S is a subset of the transversals of the three lines that contain the segments. It is a well known fact that the transversals of three lines determine a ruled quadric [14] and therefore, any forbidden point corresponding to three segments of S belongs to a ruled quadric. In some degenerated cases, a whole plane of forbidden points may exist. Figure 3 shows a planar configuration of four segments such that any line that cuts the "central" segment also cuts two of the other three segments. This violates condition 3 of regularity, so the supporting plane of the four segments is a whole plane of forbidden points.

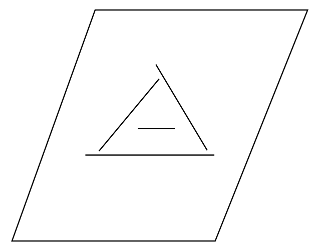

FIG. 3. *Four coplanar segments that determine a whole plane of forbidden points*

To sum up, the forbidden points of a set of segments either belong to the $\binom{n}{2} \in O(n^2)$ planes that contain the wedges, or belong to the $\binom{n}{3} \in O(n^3)$ ruled quadrics given by the transversals to the three straight lines that contain any three segments of S. Since the planes and the quadrics are finite in number and have zero measure in space, we conclude that it is always possible to find a regular perspective projection. We thus establish the following:

OBSERVATION 3.2. *There always exists a regular perspective projection of a set of disjoint segments in space.*

Let us see how to compute a regular projection of a set of n disjoint segments of S. First we translate the coordinate system $oxyz$ to $cxyz$ in such a way that all endpoints of segments have coordinates strictly positive and no plane defined by c and a segment of S contains the axis cx. All this can be done in $O(n)$ time. If c is not forbidden we are done. Otherwise, as cx is not by construction a line of forbidden points, either c is the only forbidden point of cx, and we can pick any other point from cx, or we can select any point interior to the segment ci, where i is the forbidden point on cx closest to c, but different from c.

In order to avoid the forbidden situations due to regularity conditions 1 and 2, we can use a brute force algorithm. We compute the intersection of each of the $O(n)$ lines that contain the segments and each of the $O(n^2)$ double wedges of forbidden points with the cx axis; finally, we choose among the intersection points the point i closest to c and not equal to c. If we also used a brute force algorithm to avoid the forbidden situations due to regularity condition 3 , then we would obtain an algorithm that would run in $O(n^3)$ time. The key to decrease this complexity is to employ a property of continuity of the arrangement of segments $\mathcal{A}(S^*)$, where S^* denotes the projection from c of the segments onto the plane $y = -1$. In the worst case, the construction of $\mathcal{A}(S^*)$ requires $O(n^2)$ time and space. If when moving continuously along the cx axis, beginning at c, the forbidden point first found corresponds to a forbidden situation due to condition 3, by continuity it will be caused by three segments of S that are projected on a triangular cell of $\mathcal{A}(S^*)$. Hence, it suffices to compute the $O(n^2)$ quadrics determined by three segments of S whose projections are triangular cells in $\mathcal{A}(S^*)$ and then, if it exists, to search for the intersection point $i, i \neq c$, of the quadrics with the cx axis closest to c.

The algorithm described above has time complexity $O(n^2)$: computing c and deciding if c defines a regular perspective projection of S takes linear time and space; constructing $\mathcal{A}(S^*)$ and searching in $\mathcal{A}(S^*)$ takes quadratic time and space. This leads to the following:

THEOREM 3.2. *Given a set of n disjoint segments in space, a regular perspective projection can be obtained in $O(n^2)$ time and space.*

When c is a regular projection center, it may be desirable that, when moving c by a small amount in space, the new perspective projection of S remains regular. We call the projection center of *maximum tolerance*, the point c_t that allows maximum movement in space in such a way that the projection of S is always regular. In the general case, the time complexity of the algorithm that finds the center of maximum tolerance is high, but this time complexity decreases when we restrict the possible position of the center of projection. A frequent case is when the projection center lies on a line segment. In this situation we have the following:

THEOREM 3.3. *Given a set of n disjoint segments in space, if the projection center of maximum tolerance that lies in a segment of a line exists, it can be found in $O(n^3 \log n)$ time and $O(n^3)$ space.*

Proof. We can decide in $O(n)$ time if the line segment l consists of forbidden points. If not all the points of l are forbidden points, then the intersection of l with the $O(n^2)$ double wedges and the $O(n^3)$ quadrics of forbidden points will generate a set L of $O(n^3)$ forbidden points of l. We can sort the points of L in $O(n^3 \log n)$ time and then, searching in order the points of L in $O(n^3)$ time, find the segment s_t of maximum length determined by two consecutive ordered points of L. The projection center of maximum tolerance that lies in l is the mid-point c_t of the segment s_t. ■

The definition of regular perspective projection of a set of disjoint segments can be extended to $3D$ simple polygonal chains and $3D$ wire-frame drawings of graphs, i.e. $3D$ Fáry drawings in which all edges are line segments with disjoint interiors. The fundamental difference from the regular perspective projections of a set of disjoint segments is that since two consecutive edges of a simple polygonal chain or a Fáry drawing share a vertex, the same occurs with their projection. The forbidden points will be those determined by the edges of the simple polygonal chain or the Fáry drawing considered as a set of segments, with the exception of the points of the lines that pass through a vertex and that do not cut any other edge distinct from the two consecutive edges that share the vertex. Therefore, from the former observations and the analysis of the methods used above, it follows that the previous theorems can be easily extended to a simple polygonal chain and to a Fáry drawing in space.

In the work of Bhattacharya and Rosenfeld [6], a special type of regular orthogonal projections of a polygonal chain, called *Wirtinger* projections, is introduced. In a Wirtinger projection, no two consecutive edges of the simple polygonal chain can be projected colinearly. To the forbidden points determined by the simple polygonal chain, we must add the points of the planes that contain two consecutive edges of the chain. Since the planes have zero measure in space, the results obtained for the regular perspective projections of a simple polygonal chain remain valid for the Wirtinger perspective projections.

4. MINIMUM-CROSSING AND SIMPLE PROJECTIONS

DEFINITION 4.1. Let S be a set of disjoint segments in space. The perspective projection of S is said to be **minimum-crossing** when the number of crossings of the projected segments of S is minimum. If the minimum number of crossings is zero we call such projections **simple** or **crossing free**.

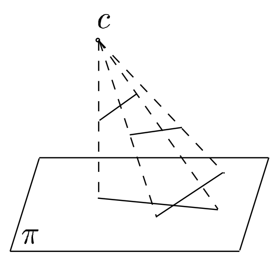

FIG. 4. *Non-simple segment projection*

As in the case of regular projections, the number of crossings depends only on the position of the projection center with respect to the segments of S.

THEOREM 4.1. *Given a set of n disjoint segments in space, deciding whether their projection is simple can be done in $O(n \log n)$ time and $O(n)$ space.*

Proof. It suffices to decide if two of the n projected segments of S^* intersect. To do this, we can use the algorithm described in [15] that detects if any two segments intersects in $O(n \log n)$ time and $O(n)$ space. ■

When the projection center c belongs to a transversal of two segments of S the point c is a forbidden one. Let $T(s, s')$ be the set of transversals of the segments $s, s' \in S$. Since $T(s, s')$ is a region of space that does not have zero measure (see Figure 5), it is possible that a simple perspective projection of S does not exist. The

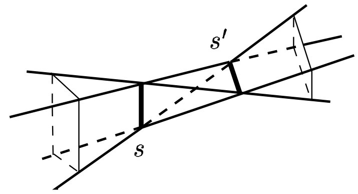

FIG. 5. *Forbidden points of two segments*

problem of deciding if a simple perspective projection of S exists can be transformed to the problem of deciding if the $O(n^2)$ regions determined by every two segments cover the space. We will denote by T the contour of the union of those regions. Therefore, a simple perspective projection of S exists if and only if T does not cover the space. In this case, if we take any point c in the complement of T as the projection center, we will obtain a simple perspective projection of S.

THEOREM 4.2. *Given a set of n disjoint segments in space, finding a minimum-crossing perspective projection can be done in $O(n^6)$ time. When the minimum number of crossings is zero the perspective projection found is simple.*

Proof. When the projection of two segments s and s' of S cross each other, then the projection center c belongs to the region $T(s, s')$. Therefore, in order to determine the center of projection c of the minimum-crossing perspective projection of S, we can choose a point c covered by the minimum number of such regions. Let P be the set of the $O(n^2)$ planes determined by the segments of S taken two by two. Let $\mathcal{A}(P)$ be the arrangement corresponding to the planes of P. It suffices to compute the arrangement $\mathcal{A}(P)$ using the algorithm of [16], and during the construction, for each cell of the arrangement to compute the $T(s, s')$ regions that contain it. The entire procedure can be done in $O(n^6)$ time and space. By doing depth-first search on the dual graph of $\mathcal{A}(P)$ we can compute in $O(n^6)$ time the cell of $\mathcal{A}(P)$ covered by the minimum number of $T(s, s')$ regions. ■

The time complexity of finding a minimum-crossing perspective projection decreases when we restrict the possible positions of the center of projection.

THEOREM 4.3. *Given a set of n disjoint segments in space, finding a minimum-crossing perspective projection, with the restriction that the projection center lies on a line segment, can be done in $O(n^2 \log n)$ time and $O(n^2)$ space.*

Proof. The points of the segment l covered by the minimum number of regions $T(s, s')$ produce perspective projections of S with the minimum number of crossings. The intersection of l with each $T(s, s')$ gives one or two sub-segments of l. When we compute the intersection of l with the $O(n^2)$ regions $T(s, s')$, we obtain a set L of $O(n^2)$ sub-segments in $O(n^2)$ time. In order to find the points of l covered by the minimum number of $T(s, s')$ regions, we can sort the set E of endpoints of the segments of L in $O(n^2 \log n)$ time and then, by traversing the $O(n^2)$ points of E in order, find the parts of the sub-segments of L covered by the minimum number of $T(s, s')$ regions. ■

The definition of minimun-crossing perspective projection of a simple polygonal chain or a Fáry drawing in space is analogous to the definition for a set of disjoint segments in space. Analyzing the techniques used in the study of the minimun-crossing perspective projections of a set of disjoint segments, it is evident that the results we have obtained for these, are also valid for simple polygonal chains and $3D$ Fáry drawings.

5. MONOTONIC PROJECTIONS

5.1. Geometric preliminaries

We recall some definitions to be used later about *Plücker coordinates and coefficients*. We refer the interested reader to [17, 18] for further details on this topic. Frequently, it is useful to represent an oriented line by its Plücker coordinates and coefficients. The reason for this is that they allow us to represent an oriented line as a point and a hyperplane in the oriented projective space of dimension five P^5, in such a way that we can solve problems related to oriented lines by using known results about hyperplane arrangements in P^5. From now on, $l = ab$ will denote the oriented line from a to b in that order. Let $a = (a_0, a_1, a_2, a_3)$ and $b = (b_0, b_1, b_2, b_3)$, with $a_0, b_0 > 0$, denote the homogeneous coordinates of a and b, respectively. By definition, the Plücker coordinates of l are the six-tuple $\pi(l) = (\pi_{01}, \pi_{02}, \pi_{12}, \pi_{03}, \pi_{13}, \pi_{23})$ with $\pi_{ij} = a_i b_j - a_j b_i, \quad 0 \leq i < j \leq 3$. The Plücker coordinates of l can be interpreted as the homogeneous coordinates of a point in P^5. We call $\pi(l)$ the *Plücker point* of l in P^5. The Plücker coefficients of l are defined by the six-tuple $\omega(l) = (\pi_{23}, -\pi_{13}, \pi_{03}, \pi_{12}, -\pi_{02}, \pi_{01})$. The Plücker coefficients of l can be thought of as the coefficients of the hyperplane $h(l) = \{p \in P^5 \,|\, \omega(l) \cdot p = 0\}$ in P^5, referred to as the *Plücker hyperplane* of l. The Plücker hyperplane $h(l)$ of l induces open positive and negative Plücker half-spaces given by $h^+(l) = \{p \in P^5 \,|\, \omega(l) \cdot p > 0\}$ and $h^-(l) = \{p \in P^5 \,|\, \omega(l) \cdot p < 0\}$.

Let $l = ab$ and $l' = cd$ be two oriented lines. When $\pi(l) \in h(l')$, then $side(l, l') = \pi_{01}(l)\pi_{23}(l') - \pi_{02}(l)\pi_{13}(l') + \pi_{12}(l)\pi_{03}(l') + \pi_{03}(l)\pi_{12}(l') - \pi_{13}(l)\pi_{02}(l') + \pi_{23}(l)\pi_{01}(l') = 0$. In general, the absolute value of $side(l, l')$ is six times the volume of the tetrahedron $abcd$. Therefore, lines l and l' are coplanar, incident or parallel, if and only if $side(l, l') = 0$. The sign of $side(l, l')$ gives the orientation of the tetrahedron $abcd$ (that can be interpreted with the right-hand rule) and allows us to define the relative orientation of the two oriented lines l and l'. When $side(l, l') > 0$, then it is $\pi(l) \in h^+(l')$ and the line l' is located to the right of the line l. Similarly, if $side(l, l') < 0$, then it is $\pi(l) \in h^-(l')$ and the line l' is located to the left of the line l.

Note that the six-tuple $\pi = (\pi_{01}, \pi_{02}, \pi_{12}, \pi_{03}, \pi_{13}, \pi_{23})$ represents an oriented line if and only if it satisfies the quadratic relation $\pi_{01}\pi_{23} - \pi_{02}\pi_{13} + \pi_{12}\pi_{03} = 0$, which states that every line is incident to itself. The former quadratic relation defines a four-dimensional subset of P^5 of degree two, referred to as the *Plücker hypersurface* Π.

5.2. Monotonicity in space

Given a line l, the pencil of planes that goes trough l will be denoted by $H(l)$.

Definition 5.1. A simple polygonal chain in space is said to be **strictly monotonic** with respect to a line l when the intersection of the polygonal chain with any plane of $H(l)$ is the empty set or a point.

Observe that if P is strictly monotonic with respect to line l, then l can not intersect the convex hull of the chain P, because on the contrary there always exist a plane of $H(l)$ that intersects P in two points. Note that when we move the line to infinity, we obtain the standard definition of monotonicity of a polygonal chain with respect to a direction in space: the intersection of the simple polygonal chain with any plane of the pencil of parallel planes orthogonal to the fixed direction is the empty set or a point.

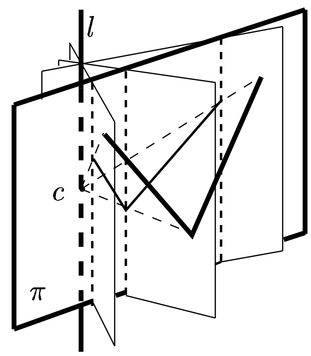

FIG. 6. *Polygonal chain strictly monotonic with respect to a line*

Definition 5.2. Let P be a simple polygonal chain in space. A perspective projection P^* of P is said to be **monotonic** when the polygonal chain P^* is strictly monotonic with respect to some direction of the projection plane.

Let P be a strictly monotonic polygonal chain in space with respect to an oriented line l. Let c be a point of l and let π be a plane that separates l and P. Denote by P^* the projection of P from center c onto plane π. Finally, let $\vec{\pi}$ be a vector of the plane π orthogonal to the direction vector of the straight line l. Under these conditions the projection P^* of P is a monotonic projection. More specifically, we have the following:

Theorem 5.1. *The polygonal chain P^* of the plane π is strictly monotonic with respect to the direction $\vec{\pi}$.*

Proof. We prove by contradiction that the intersection of the polygonal chain P^* with any line of the pencil of parallel lines orthogonal to the direction $\vec{\pi}$ is the empty set or a point. Assume that a line r in π orthogonal to $\vec{\pi}$ intersects two edges of P^* at the points e^* of and f^* respectively. Let e and f be, respectively, the points of P whose projections are e^* and f^*. Under these conditions, the plane $\pi(r)$ in $H(l)$ that passes through r intersects the simple polygonal chain P in e and f. Therefore the chain P is not strictly monotonic with respect to the line l contradicting our hypothesis. ■

Let $P = \{v_1, \cdots, v_n\}$ be a simple polygonal chain in space. For $i = 1, \cdots, n-1$, let l_i be the oriented line going through the vertices v_i and v_{i+1} in that order. Let:

$$L = \bigcap_{i=1}^{n-1} h^+(l_i) \quad ; \qquad R = \bigcap_{i=1}^{n-1} h^-(l_i) .$$

We will denote $S(P)$ the set of Plücker points $\pi(s)$ of lines s that intersect with the convex hull $conv(P)$ of the simple polygonal chain P.

THEOREM 5.2. *The simple polygonal chain P is strictly monotonic with respect to line l if and only if $\pi(l) \in (L \cup R - S(P)) \cap \Pi$.*

Proof. If P is strictly monotonic with respect to l, then we know that $\pi(l) \notin S(P)$. Besides, it must be $\pi(l) \in (L \cup R) \cap \Pi$ because, on the contrary, line l is located in different sides of two consecutive lines l_i and l_{i+1} and, consequently, there exist a plane of $H(l)$ that intersects l_i and l_{i+1}, in contradiction with the monotonicity of P with respect to l. Conversely, if $\pi(l) \in (L \cup R - S(P)) \cap \Pi$, then P is strictly monotonic with respect to l. Indeed, if we do a rotational sweep of a plane that passes consecutively through l and through the points of the chain P, from v_1 to v_n, then the sweep plane and P always intersect at a point. ■

THEOREM 5.3. *Given a simple polygonal chain of n vertices, determining whether it is strictly monotonic with respect to some line can be done in $O(n \log n)$ time.*

Proof. We will denote $L' = L - S(P)$ and $R' = R - S(P)$. In order to determine whether P is strictly monotonic respect to some line, it suffices to check whether $L' \cap \Pi \neq \emptyset$ or $R' \cap \Pi \neq \emptyset$. We begin by computing, in $O(n \log n)$ time, the convex hull $conv(P)$ of P and then constructing the set $OH(P)$ of oriented lines, according to the order of the vertices of P, contained in the edges of $conv(P) - P$. Since L' (R') is the set of lines located to the left (right) of all the lines l_i that does not intersect $conv(P)$, then these lines must be located also to the left (right) of the lines of $OH(P)$. So, we have:

$$L' = \left(\bigcap_{i=1}^{n-1} h^+(l_i)\right) \bigcap \left(\bigcap_{s \in OH(P)} h^+(s)\right) \; ; \; R' = \left(\bigcap_{i=1}^{n-1} h^-(l_i)\right) \bigcap \left(\bigcap_{s \in OH(P)} h^-(s)\right) .$$

Since L' and R' are the intersection of halfspaces, they are convex polytopes, possibly unbounded, of P^5. Note that because Π is of degree two, the number of intersection points with each face of any dimension of L' or R' is constant. Therefore, it suffices to determine if L' or R' are empty sets. This can be done in $O(n)$ time using linear programming techniques [19]. ■

Given the simple polygonal chain P of n vertices, we will find all the lines c such that P is strictly monotonic. Note that it is possible that any such line does not exist. Transferring the problem to the P^5 projective space, it is equivalent to computing $(L' \cap \Pi) \cup (R' \cap \Pi) = (L' \cup R') \cap \Pi$, a set that eventually can be empty. We will follow the method described in [17] in order to compute all the features of a polytope. First of all we must have a bound on the number of the features of the polytope; by the Upper Bound Theorem [20], the number of features of the polytopes L' and R' is known to be $O(n^{\lfloor 5/2 \rfloor}) = O(n^2)$. It is not difficult to find configurations that attain this bound. Since Π is of degree two, intersecting a polytope with Π can only increase the above complexity by a constant factor. Finally, in order to compute the polytopes L' and R' we can use an algorithm of Chazelle [21] that constructs the representation of a polytope by the incidence graph of their faces [20] in $O(n^2)$ time. Therefore, we have the following:

THEOREM 5.4. *Given a simple polygonal chain of n vertices, computing all the lines with respect to which it is strictly monotonic can be done in $O(n^2)$ time.*

Suppose that $L' \cap \Pi$ and $R' \cap \Pi$ are empty. If a chain is not strictly monotonic with respect to a line, it may be desirable to know which lines are closer to make the chain strictly monotonic (in other words, to compute the set of lines that lies to the left or to the right of the maximum number of edges of P).

THEOREM 5.5. *Given a simple polygonal chain of n vertices, computing the set of lines that lies to the left or to the right of the maximum number of edges of the polygonal chain can be done in $O(n^5)$ time. When this maximum number equals n, the polygonal chain is strictly monotonic with respect to the lines found.*

Proof. First, we construct the arrangement $\mathcal{A}(H)$ of the set of $O(n)$ hyperplanes determined by the edges of the polygonal chain P. This process can be done in $O(n^5)$ time using the method of [16]. Next we determine, in $O(n^5)$ time, the cells of $\mathcal{A}(H)$ that intersect Π and, by selecting a point for each of these cells, we compute the number of times that the line that represents the point lies to the left and to the right of the edges of P. Finally, by doing depth-first search on the dual graph of $\mathcal{A}(H)$, we compute in $O(n^5)$ time the cells of $\mathcal{A}(H)$ formed by points that represent lines that lie to the left or to the right of the maximum number of edges of P. ■

The definition of monotonicity of a simple polygonal chain can be easily extended to a geometric rooted tree (in [3] several applications of geometric rooted trees to Medicine are illustrated). A geometric rooted tree is strictly monotonic with respect to a line when the path from the root to every vertex is a monotonic polygonal chain with respect to the given line. So, it is evident at once that the results that we have obtained for polygonal chains are also valid for rooted trees. Monotonic projections are also important when viewing hierarchical graph drawings.

6. CONCLUSIONS

In the paper by Bose et al. [3], the term nice projection was coined for those projections which preserve certain properties of the objects in space. In their work, they consider orthogonal projections and define several criteria of niceness. Here we consider some of their criteria and use perspective projections instead. Computing nice perspective projections is harder in general because there is an additional parameter (the projection center) to be taken into account and that makes the space of forbidden directions more complex.

Algorithms have been presented to compute nice perspective projections under several measures of niceness: regularity, simplicity and minimum-crossing and monotonicity. The results presented here are immediately applicable to problems that arise in areas such as Computer Vision, Computer Graphics, Graph Drawing, Knot Theory and Computational Geometry.

Almost all the complexity results discussed in this paper are restricted to inputs of idealized points and line segments viewed under the real RAM model of computation. This may be appropriate for some areas such as Knot Theory. For other areas, such as Computer Graphics and Computer Vision, such an assumption implies infinite resolution images, an unrealistic assumption. Extending our work to projections of $3D$ objects for a *finite resolution* image model is an open problem.

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