Generating “Good” Musical Rhythms Algorithmically

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Abstract

While it is difficult to define precisely what makes a “good” rhythm good, it is not hard to specify properties that contribute to a rhythm’s goodness. One such property is that the mirror-symmetric transformation of the rhythm about some axis of the rhythm’s cycle, represented as a circle, be the same as its complementary rhythm. Rhythms that have this property are called *interlocking reflection* rhythms. Another family of rhythms termed *toggle* rhythms are those cyclic rhythms that when played using the *alternating-hands* method, have their onsets in one cycle divided into two consecutive sets such that first set is played consecutively with one hand, and the second set is played consecutively with the other hand. Several simple rhythm-generation methods that yield good rhythm timelines with these properties are presented and illustrated with examples.

1. Introduction

It is well known that music and mathematics complement each other in wonderful and useful ways [1]-[4], [7]-[10], [25]. For example, it has recently been shown that a simple algorithm dating back to Euclid of Alexandria may be used to generate most of the traditional rhythms used in music throughout the world [2]. The algorithm in question is one of the oldest and well-known algorithms, described in Propositions 1 and 2 of *Book VII* of *The Elements*, and today referred to as the *Euclidean algorithm*. This algorithm computes the greatest common divisor of two given natural numbers. The computer scientist Donald Knuth calls it the “granddaddy of all algorithms, because it is the oldest nontrivial algorithm that has survived to the present day.” The idea is captivatingly simple. Repeatedly replace the larger of the two numbers by their difference until both are equal. This last number is then the greatest common divisor. In [1] it is shown how the execution of the algorithm, or more precisely the algorithm’s history during execution, generates the rhythms. This algorithm then implicitly captures a fragment of the notion of what makes a “good” rhythm good. Characterizing such rhythms is the ultimate, but not easily attainable, goal of this research. While it is difficult to define precisely the necessary and sufficient conditions that make a “good” rhythm good, it is not too hard to list a variety of properties that contribute to a rhythm’s goodness. One such property is that the mirror-symmetric transformation of a rhythm about some axis of the rhythm cycle, represented as a circle, be equal to its complementary rhythm. Rhythms that have this property will be called *interlocking reflection* rhythms. Another family of rhythms termed *toggle* rhythms consists of those cyclic rhythms that when played using the alternating-hands method, have their onsets in one cycle divided into two consecutive sets such that first set is played consecutively with one hand, and subsequently the second set is played consecutively with the other hand. Thus it is as if a toggle switch at one point during the execution of the rhythm simultaneously turns off the right hand and turns on the left hand, and at a later point switches the roles of the hands back to their starting positions. Of course any rhythm whatsoever has the toggle property if we choose to ignore the fact that toggling, by definition, may be done only twice. Strictly speaking the rhythms defined here should be called *two-state-left-right toggle* rhythms, or *linearly separable toggle* rhythms, as will become clear in the following. However, for convenience the shorter name *toggle* rhythms will be employed in this work.

This paper describes several simple rhythm-generation methods that yield rhythms with the properties mentioned in the preceding. These properties are closely related to the more general notion of *syncopation*. Syncopation leaves out stresses or adds stresses where they are unexpected. Much has been written about how adding syncopation to a perfectly regular rhythm may convert a monotonous dull rhythm into a "good" one. According to George Coleman Gow, "Syncopation is the piquant of rhythm." [13, p. 649]. Gow also quotes C. Abdy Williams on p. 648:

> "Though a hard and fast line cannot be drawn, it may be said in a general way that when rhythms begin to omit any of their accents they begin to appeal to the imagination and the intellect more than to the physical faculties. For it requires a higher degree of culture to recognize a thing that is only hinted at than a thing that is plainly set before one."

The methods considered here are radically different from those used for generating rhythms automatically, described in the artificial intelligence and music information retrieval literatures. These latter approaches are inspired either by models of biological processes such as neural networks that learn from experience, by genetic programming that models the evolutionary laws of natural selection, or by statistical models such as Markov processes [26]. Many of these techniques are based on guided random search of the space of all possible rhythms [11], [12], [14]. Typically genetic methods first define a measure of rhythmic "goodness" generally termed a *fitness function*, and then use simple rules for transforming a given collection of rhythms in such a way as to improve their fitness. These rules are usually stated in general terms as *reproduction*, *crossover*, and *mutation*, applied in this order. Reproduction selects a pair of rhythms, say A and B, at random from the collection. Crossover involves creating new offspring rhythms of A and B by swapping some elements from A to B and vice-versa. Mutation involves changing one of the elements of a new offspring of A or B at random, and with low probability. Finally, the algorithm stops (or is stopped) when the fitness function has (or seems to have) reached a maximum value [11].

Gibson and Byrne [12] incorporate a neural network in their genetic approach. First they use humans to label a collection of training-data rhythms as either "good" or "bad." Then they use the trained neural network to classify rhythms generated by the genetic algorithm as either "good" or "bad," thus serving as the fitness function. Damon Horowitz [14] describes an interactive approach that allows the user to "simply execute fitness functions (that is, to choose which rhythms or features of rhythms the user likes) without necessarily understanding the details or parameters of these functions." This "ostrich head-in-the-sand" approach may be attractive and useful to those composers and other users that are satisfied with only the end product. By contrast, the methods proposed here for generating "good" rhythms are *structural* in nature, and guided by musicological and empirical knowledge of rhythms that humanity has come to cherish over thousands, if not millions, of years of evolution. The crux in these methods is precisely the understanding of the details, and the elimination of the parameters in neural networks that must be tweaked in order to obtain good rhythms. These methods are closer in spirit to computational music theory, and represent an attempt to understand the temporal structures that make a rhythm "good." Furthermore, if desired the properties discussed here may also be incorporated into fitness functions for use in the standard genetic algorithms.

The music theory literature contains some work on the related topic of rhythmic consonance and dissonance [15]. However, these notions refer more to the description of a pair of rhythms rather than to a property of a single rhythm, and there is no attempt to associate consonant with "good" or dissonant with "bad." Two rhythms that differ considerably in their beat structure, such as the two-against-three, or three-against-four beats are considered dissonant [15], yet in the cultural context of West and Central Africa, rhythms that combine all three rhythms are part of the standard repertoire of polyrhythmic rhythms [18]. A rhythm that sounds dissonant, complex, or new to one culture may be the order of the day to another, as was recounted more than one hundred years ago, using some language that is now considered culturally offensive, by Thomas F. Dunhill following one of his travels in the South Pacific [16, p. 4].

> "One of the most interesting of my own-experiences during one rather extensive travels of two years ago, was a brief visit to one of the islands of Samoa, in the Pacific, where I had the great pleasure of hearing a really characteristic song sung by the natives. It consisted of a short phrase of two bars repeated many times over with growing intensity. The fragment was peculiar and fascinating, and was very decidedly cast in a rhythm of five beats to the bar. This circumstance struck me very much, because, when a certain celebrated modern composer wrote the whole of a symphonic movement in that time it was generally regarded, in spite of some established precedence, as a new evidence of some fresh possibilities in rhythm. Yet here were the far-away, primitive Pacific islanders singing a tune in a similar measure, naturally, persistently, and with no evident consciousness of its unusual pulsation."

The notion of what is a "good" rhythm may also change with time and the whims of fashion. Quintuple rhythms (those that contain five beats to the bar, in Dunhill's terminology), were used extensively by the ancient Greeks, especially in religious ceremonies, and in tragic plays as a means of expressing intense emotions. With the advent of Christianity they almost disappeared in Europe for two thousand years, before experiencing their renaissance at the turn of the 20th Century [17]. On the other hand, in folk music from many cultures quintuple rhythms remained a stable part of its repertoire [38].

Some attempts have been made to explain the psychological efficacy of certain well-known good rhythms such as the *clave son* using psychological principles of Gestalt perception, and then translating these principles to geometric properties [19], [21]. While it is no clear how to convert these principles to algorithms that will generate other "good" rhythms, the work in [19] comes closest to that discussed here.

The structural approach to characterizing "good" rhythms, that is the main focus of this paper forms part of a broader and much more ambitious goal to identify *universal* "good" rhythms, that is, rhythms that are considered "good" across all cultures and all time periods, and to explain their saliency, if possible using the language of mathematics.

Throughout this paper rhythms are represented either as periodic binary sequences using box notation, or by mapping the period onto a circle of time [4], [43]. For example, Figure 1 (a) shows a sequence of 16 boxes all of which are filled with a black disk. Each box represents an equal unit of time, and if it is filled it is sounded, otherwise it is silent. In Figure 1 (a) all the boxes are filled, thus creating a rather humdrum uniform stream of sounds. In the words of Roe [27] "A sound uniformly continued, or uniformly repeated, is uninteresting." Indeed, most musicologists would not even consider such a stream as a *bona fide* rhythm. Simha Arom calls such a pattern a *metric continuum* [18]. Figure 2 (right) shows the rhythm of Figure 1 (d) represented as a clock diagram.

By this definition the entire family of rhythms is astronomically large. For example, even the rhythms consisting of only 16 boxes, 5 of which are filled, number 4,368. Of course many of these rhythms would not be considered "good." Here the focus is on a much smaller subset of rhythms called *timelines* [20], [22], [23], [24]. For Kofi Agawu [20, p. 1] a timeline "also called bell pattern, bell rhythm, guideline, time keeper, topos, and phrasing referent—is a distinctly shaped and often memorable rhythmic figure of modest duration that is played as an ostinato throughout a given dance composition." Agawu provides an in-depth analysis of timelines in both, cultural and structural contexts, focusing primarily on a timeline often called the "standard" pattern [20]. He also provides a top-down, hierarchical, four-step procedure for the generation of salient timelines, which reflects the African musical mind. The algorithms presented here, while purely mathematical in their execution, and thus quite different in *process* from Agawu's cultural-practice based approach, nevertheless yield timelines that may be readily obtained using his 4-step procedure, and thus both approaches are similar with respect to the end *product*.

2. Reflection Rhythms

One property of "good" rhythms in general, and timelines in particular, is that the mirror symmetry of a rhythm about some axis is equal to its complementary rhythm. Rhythms that have this property are called *interlocking reflection* rhythms. This section presents via examples, two simple rhythm generation methods that yield rhythms with this property: the *paradiddle* algorithm and the *alternating-hands* algorithm.

Paradiddle Method: To illustrate this method consider a time span of 16 pulses, and refer to Figure 1. First, insert an onset at every pulse in the cycle of the rhythm, as in (a), where the label 'Right' indicates the rhythm is played with the right hand, and the label 'period' indicates the length of the rhythm that repeats itself. If we denote by the symbols R and L the striking of the drum with the right and left hand respectively, then this rhythm repeats the R symbol 16 times. Next, transform this rhythm into a rhythm for two hands by alternating the onsets between the left and right hands, preferably each hand playing on a different drum, as in (b). The idea here is that the right and left hands should produce different sounds either in pitch or timbre so that the listener may perceive both the right-hand and left-hand rhythms simultaneously. Note that this operation doubles the length of the period of the rhythms played by each hand. The rhythm thus repeats the RL pattern 8 times. The next step in the process involves repeating the symbol R after one instance of the RLR pattern to obtain the string RLRR, as in (c). Note that this operation doubles the period again to a length of 4. The final step involves alternating between the RLRR pattern and its mirror image LRLL, as in (d), to obtain the pattern RLRRLRLL. This operation again doubles the period of the rhythm making it have a time-span of 8 pulses. The rhythm heard on the right-hand drum has interval structure [2-1-2-3] shown in polygon notation in Figure 2 (right), with its mirror image complementary rhythm played on the left hand drum (left). Note that the rhythm [2-1-2-3] is a deep rhythm [8], has the rhythmic oddity property [39], and when played backwards becomes [3-2-1-2], which is the hand-clapping pattern in Elvis Presley's *Hound Dog*. The complete rhythm played with both hands is known as the *single paradiddle* rhythm in rudimentary snare drum technique [5], [6]. A rhythm is *deep* if each duration realized by a pair of onsets occurs a unique number of times, and a rhythm has the *rhythmic oddity* property if no pair of its onsets partitions the rhythm into two half-cycles of equal duration.

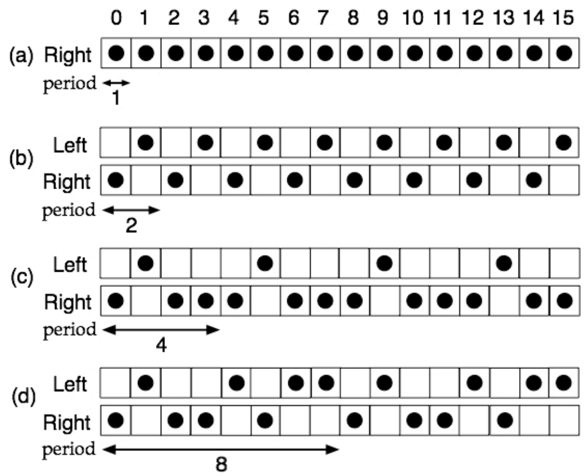

Fig. 1: Constructing a *single paradiddle* reflection rhythm.

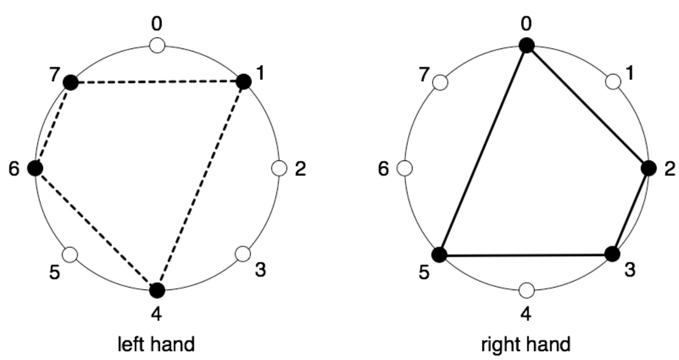

Figure 2: The left and right hand patterns of the *single paradiddle* drum rhythm.

For a second example of this method of generating good rhythms consider the construction of a ternary rhythm with 6 onsets and 12 pulses. This time we start with a rhythm consisting of 24 onsets and 24 pulses as shown in Figure 3 (a). As in the previous example this rhythm is first decomposed into its alternating right and left hand onsets as in (b). The next step shown in (c) is the only step in this process that changes. Instead of repeating the symbol R after the RLR pattern, now it is repeated after the RLRLR pattern to create an RLRLRR pattern of period 6. The final step (d) is the same as before: the RLRLRR pattern is alternated with its mirror image LRLRLL to create a rhythm with period 12. The rhythm played with the right hand, shown in polygon notation in Figure 4, has inter-onset-interval vector [2-2-1-2-2-3], which has the rhythmic oddity property, is a deep rhythm, and is the *guataca* (metal hoe blade) timeline used in Cuban *batá* drumming [45: p. 35]. The entire rhythm played with both hands is called the *double paradiddle* in rudimentary snare drum technique [5], [6].

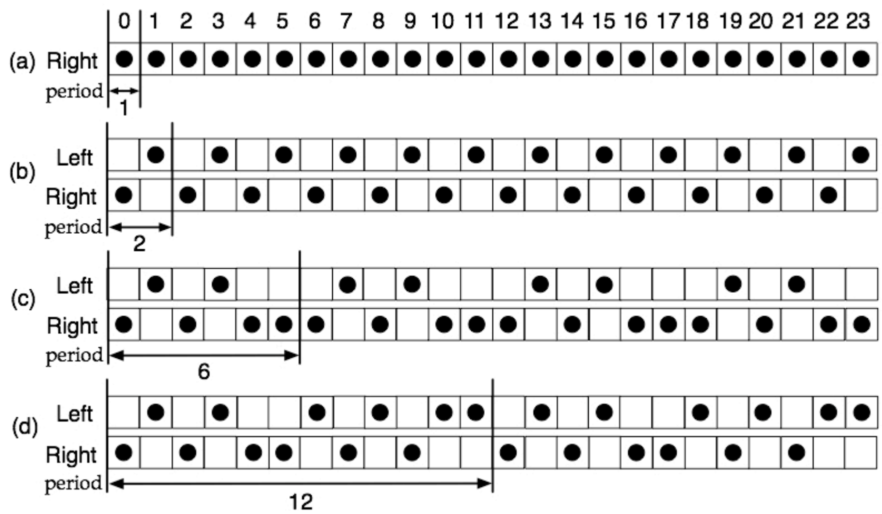

Fig. 3: Constructing a *double paradiddle* reflection rhythm.

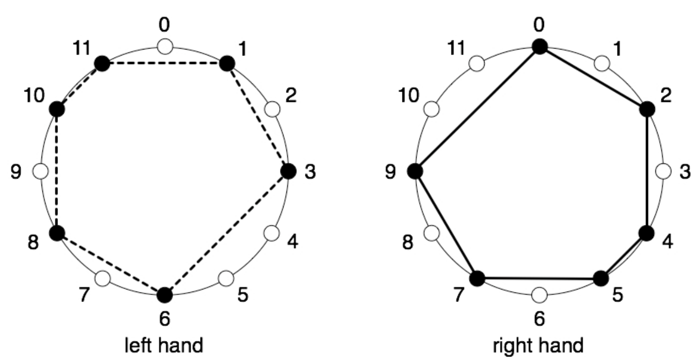

Fig. 4: The left and right hand patterns of the *double paradiddle* drum rhythm.

For the third example that illustrates this method let us generate a rhythm with 8 onsets and 16 pulses. The process for the first two steps is the same as before, but this time start with 32 pulses, as shown in Figure 5 (a) and (b). The change comes again in step (c) where instead of repeating the symbol R after the RLRLR pattern, now it is repeated after the RLRLRLR pattern to create an RLRLRLRR pattern with period 8. The final step (d) is the same as before: the RLRLRLRR pattern is alternated with its mirror image LRLRLRLL to create a rhythm with period 16. The rhythm played with the right hand, shown in polygon notation in Figure 6, has inter-onset-interval vector [2-2-2-1-2-2-2-3], which is a deep rhythm as may be seen from its interval histogram in Figure 6. This rhythm also has the rhythmic oddity property. The entire rhythm played with both hands is called the *triple paradiddle* in rudimentary snare drum technique [5], [6].

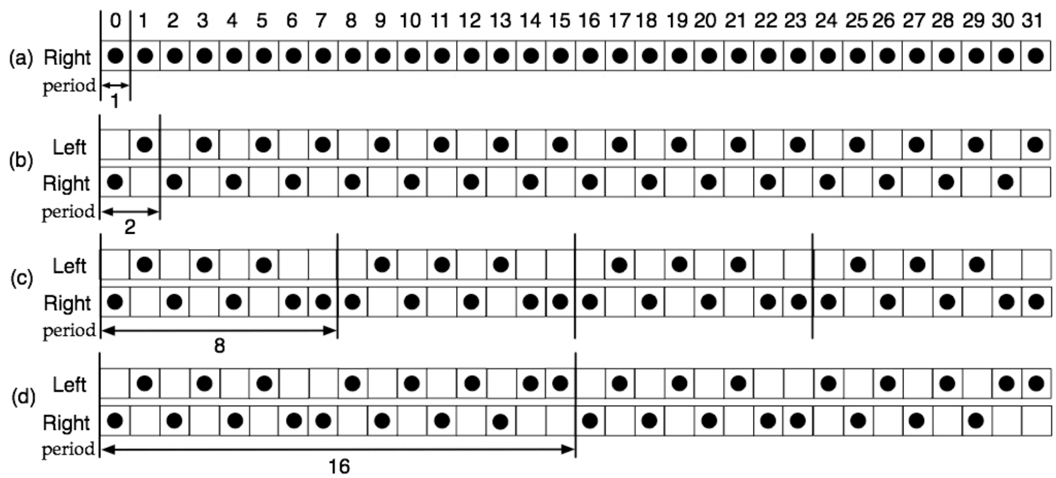

Fig. 5: Constructing a *triple paradiddle* reflection rhythm.

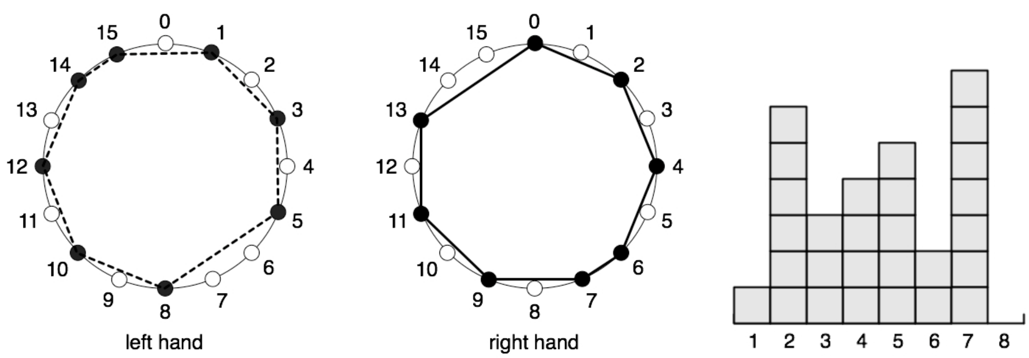

Fig. 6: The left and right hand patterns of the *triple paradiddle* drum rhythm, and their interval content histogram.

Alternating-Hands Approach: Whereas the *paradiddle method* described above puts an onset on every pulse, and then creates a pattern by breaking the alternation between the right and left hands by repeating an onset with the same hand, the alternating-hands approach maintains the alternation of right and left hands throughout the execution of the rhythm, generating a new rhythm by transforming a (usually simpler) seed rhythm. Several different transformation rules may be applied here. To illustrate one such method consider the simple 4-pulse pattern [x x x .] shown in Figure 7 (a). This rhythm is a universally used pattern dating back to at least the *ars antiqua*, associated with prosody, and known as the *short-short-long* pattern. It is also a pattern used in the *Baiaó* rhythm of Brazil, a drum rhythm in South Indian classical music, and the *polos* rhythm of

Bali. The second step of the algorithm decomposes this cyclic pattern into alternating right-hand and left-hand strokes as shown in (b). Finally, this rhythm is alternated with its mirror image, creating with the right-hand an 8-pulse rhythm that has inter-onset-interval vector [2-3-3], as shown in (c). At the same time the left hand is playing a reflected (mirror image) and translated version of this rhythm. The rhythms in (b) and (c) are typical rhythms played with the metal double castanets (see Figure 8) in the Gnawa trance music of North Africa found mainly in Morocco and Algeria. One pair of castanets is used in each hand to produce a loud and distinctive metallic clapping sound.

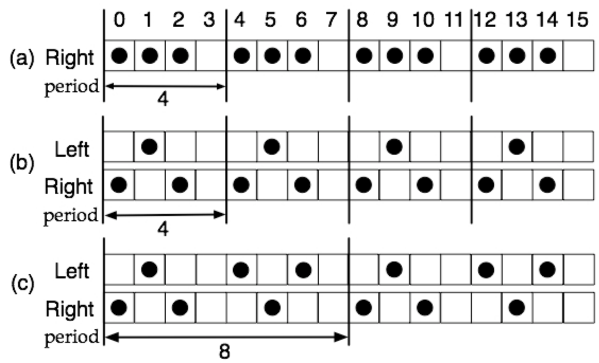

Fig. 7: Constructing the [2-3-3] reflection rhythm using the alternation method.

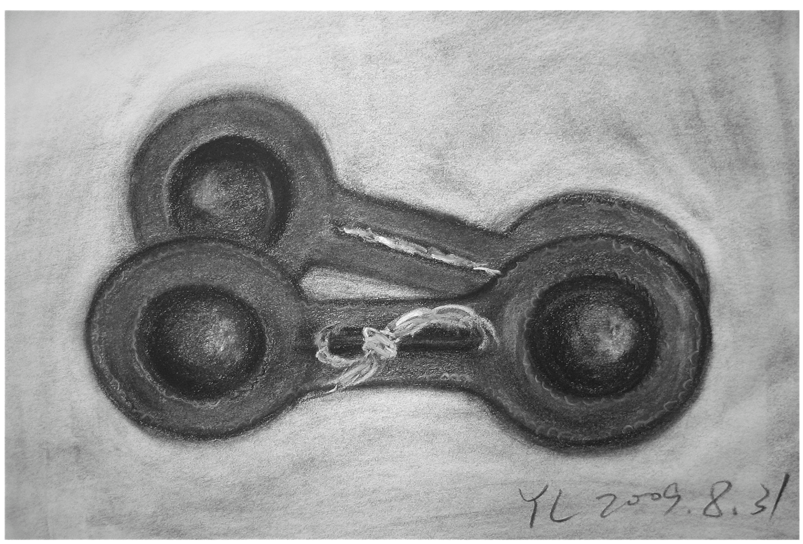

Fig. 8: Moroccan krakebs (metal double castanets). (© Yang Liu, 2009)

Figures 9, 10, and 11, illustrate how this alternating-hands method generates three of the most well known timelines used in World Music: the bossa-nova, the columbia, and the clave son, respectively. The French percussionist and teacher Vincent Manuelle explored a similar idea to develop a theory of clave rhythms. For further information about these timelines the reader is referred to [2]-[4], [37], [42].

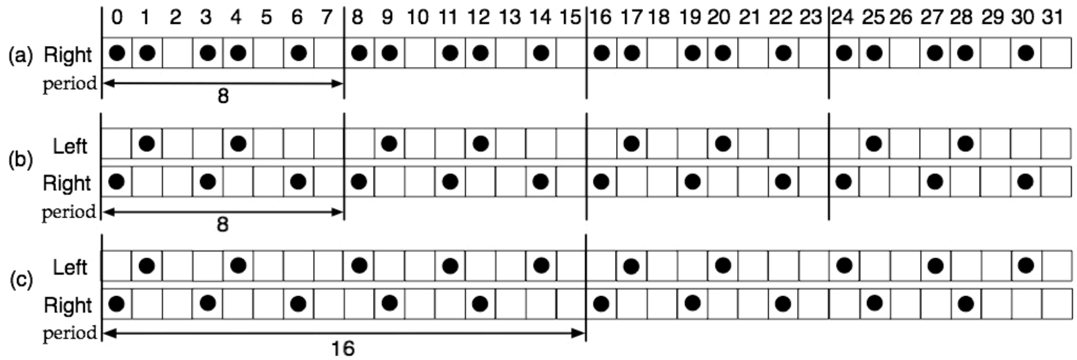

Fig. 9: Constructing the bossa-nova necklace timeline with the alternation method.

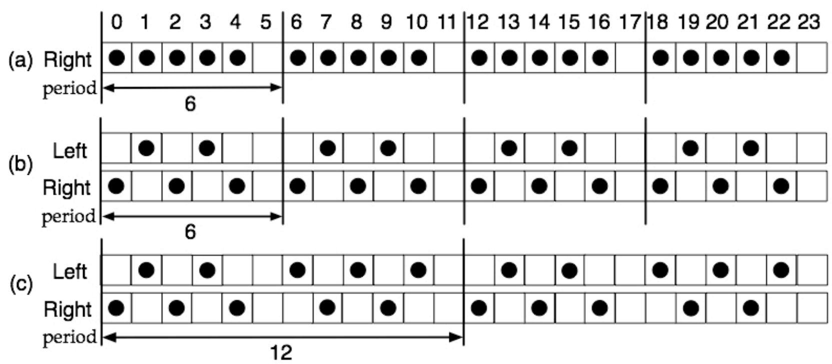

Fig. 10: Constructing the columbia timeline with the alternation method.

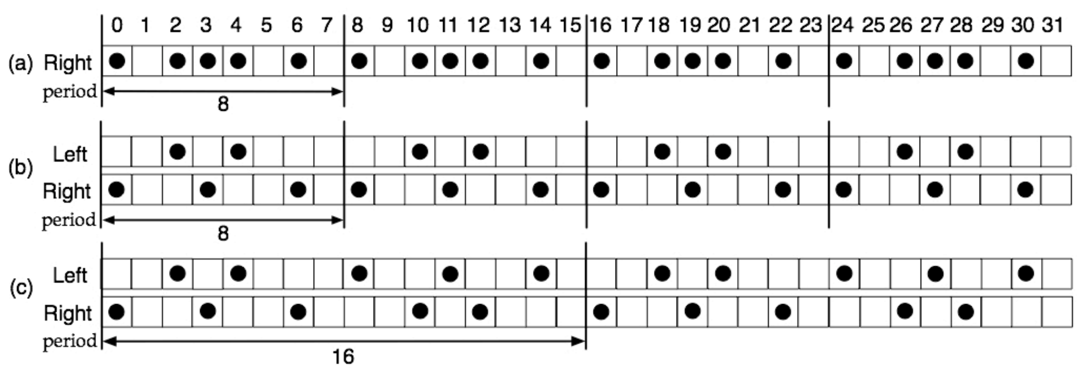

Fig. 11: Constructing the clave son with the alternation method.

3. Toggle Rhythms

In the *alternating hands* method described in the preceding section, the right and left hands continually take turns striking the instrument, much as the feet do on the ground while we walk, except that, depending on the method employed, the durations between consecutive right and left handed strokes may vary. For example, in the rhythm of Figure 7 (c) the duration between the first (right) and second (left) onsets is one pulse, but the duration between the third (right) and fourth (left) onsets lasts two pulses. In the method described in this section the rhythm emerges from the process of *accenting* the proper onsets with each hand while maintaining all the durations between left and right handed strokes equal to one pulse. In other words some strokes may be louder than others, or they may differ in timbre, or tonality. Indeed, the soft sounds may even be so muted that they are inaudible, or the hand may stop just before coming into contact with the instrument. The important point is that the motion of the hands consists of a continuous mechanical pendular alternation of the right and left hands such that all durations between adjacent pulses are equal. In other words, the downward motions of the hands realize *all* the pulses of the rhythm.

Toggle rhythms are those cyclic rhythms that when played using the alternating hands method, have their onsets divided into two consecutive sets, such that the first set is played consecutively with one hand, and subsequently the second set is played consecutively with the other hand. Thus, playing this way feels as if one hand responds to a question posed by the other hand, analogous to the customary call-and-response method of singing existent in much of Sub-Saharan Africa. The most pleasing and interesting results with this method are obtained when the left and right hands strike drums that are tuned differently, so that they produce sounds of distinct tone or timbre. However, even on a single drum the left and right hands will almost always produce distinct sounds, since they strike the drum skin at different locations, and thus the effect will still be audible and operative. However, even if all the accented strokes sound the same, the system yields good timelines. Indeed, timelines by their usual definition have the property that they do not contain accents, that is, all their onsets sound the same.

The motion of the right and left hands in this method of playing may be conveniently described with a notation such as RLRLRLRLRLRLRLRL, which indicates all the pulses present in the rhythmic cycle, as well as which pulses are struck with which hand, R standing for the right hand which strikes on even-numbered pulses, and L for the left hand which strikes on odd-numbered pulses. The rhythm that emerges from this pattern may be notated using a bold face font for the accented onsets. For example, one possible toggle rhythm with this pattern is **R**L**R**L**R**L**R**LR**L**R**L**R**L**RL. By accenting the four right-hand and three left-hand strokes, the rhythm that emerges in the form of the accented onsets may be described in box notation as [x . x . x . x . . x . x . x . .]. Note that in this example every pair of consecutive onsets played with the right hand (or left) is separated by one silent pulse. Note also that the transitions between the right-hand strokes and the left-hand strokes are separated by an interval of two silent pulses. Toggle rhythms that have this property will be called *smooth* toggle rhythms because this transition is smooth. On the other hand a pattern such as **R**L**R**L**R**L**R**L**RL**R**L**R**L**R**L**, which may be expressed in box notation by [x . x . x . x . x x . x . x . x], has no silent pulses between the transition of left-hand and right-hand pulses. This transition is abrupt or sharp, and so toggle rhythms with this property are termed *sharp* toggle rhythms.

Figure 12 illustrates an algorithm for generating a family of smooth toggle rhythms, by starting with the simplest smooth toggle rhythm that acts as a seed pattern. This seed pattern shown in (a) consists of a cycle of 8 pulses with two right-hand onsets at pulses 0 and 2, and one left-hand onset at pulse 5. This rhythm has an inter-onset-interval structure given by [2-3-3], and is used in the traditional music of several cultures: it is the bell rhythm used in the Nandon Bawaa music of the Dagarti people of Ghana [31], as well as a rhythm found in Namibia and Bulgaria [32]. From this seed pattern we may create new longer rhythms by repeatedly cutting the rhythm in half, and inserting a copy of the *left-*

right transition segment in between the resulting two pieces, as illustrated in Figure 13. The top of Figure 13 shows the 8-pulse seed rhythm with the left-right transition segment shaded. The middle shows the original rhythm cut into two equal duration pieces. Note that the cut is made at the midpoint that separates the right-hand strokes from the left-hand strokes, which in this case happens between pulses 3 and 4. The bottom shows the final 12-pulse rhythm obtained by splicing the three pieces together, which is the rhythm shown in Figure 12 (b). This process may be iterated by repeatedly inserting the shaded 4-pulse transition segment into the preceding rhythm in the same manner. In this way we may generate the remaining rhythms shown in Figure 12, as well as longer ones if desired. However, a cycle of 24 pulses is almost always long enough to serve as a timeline.

The rhythm in Figure 12 (b) with inter-onset interval structure [2-2-3-2-3], is the Fume-Fume bell pattern popular in West Africa [22], [23], and is also used in the former Yugoslavia [33]. The rhythm in (c) with inter-onset interval structure [2-2-2-3-2-2-3] is a hand-clapping timeline pattern from Ghana [21]. The rhythm in (e) consisting of 11 onsets among 24 pulses, is perhaps the longest existing smooth toggle timeline, and is played by the Aka pygmies of Central Africa [32].

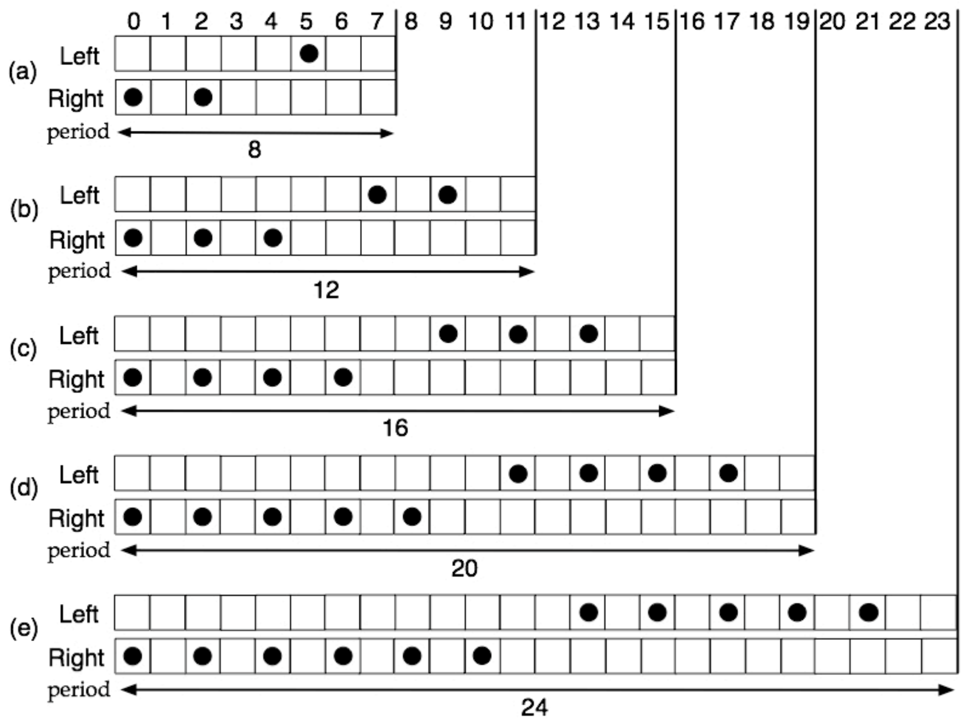

Figure 12: One method for generating smooth toggle rhythms.

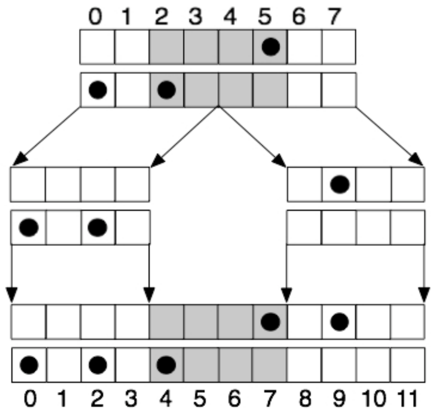

Fig. 13: Splicing a *smooth* toggle rhythm by inserting the right-left transition segment (shaded).

A similar approach may be used to generate a family of sharp toggle rhythms, as illustrated in Figures 14 and 15. Figure 14 shows a collection of six sharp toggle rhythms ranging in time spans from four to twenty-four pulses, in increments of four pulses. The top of Figure 15 shows the shaded 4-pulse right-left transition segment that must be inserted into a sharp toggle rhythm to create a new longer sharp toggle rhythm. The remainder of the figure details how the splicing may be done on the sharp toggle rhythm of Figure 14 (b). Note that here the cut made between the right-hand strokes and the left-hand strokes occurs between pulses 4 and 5, and partitions the rhythm into two unequal pieces: one with five pulses and three onsets, and the other with three pulses and two onsets. The resulting new rhythm timeline at the bottom of Figure 15 is the 7-onset 12-pulse timeline of Figure 14 (c). This is the so-called *standard pattern* used widely in Sub-Saharan Africa [22]-[24], [28]-[32]. Of course, another simple method of generating sharp toggle rhythms, if one already has the smooth toggle rhythms to begin with, is to convert a smooth version to a sharp version by adding at the appropriate places, one onset to the patterns of each hand. For example, the smooth toggle rhythm of Figure 12 (b) may be converted to the sharp toggle rhythm of Figure 14 (c) by adding one right-hand onset at pulse number 6, and one left-hand onset at pulse number 11.

One of the longest sharp toggle rhythms is the *bobanji* timeline played on a metal bell by the Aka Pygmies of Central Africa [32]; it has 13 onsets in a cycle of 24 pulses. A rotation of this timeline is shown in Figure 14 (f). The bobanji timeline is actually started on the 4th onset at pulse number 6.

The methods described in the preceding for composing smooth and sharp toggle rhythm timelines generate rhythms with the property that the first set of onsets played with the right hand has one more onset than the second set played with the left hand. This property may be easily reversed. Another set of rhythms may be determined by first interchanging the onsets played with each hand, and then playing the rhythms thus obtained in reverse order. For example such a transformation applied to the sharp toggle rhythm of Figure 14 (c) yields the rhythm **R**L**R**L**RL**R**L**R**L**. This is equivalent to starting the rhythm of Figure 14 (c) on the second onset, which in turn represents a *rotation* of the rhythm in a counterclockwise direction by a duration interval of two pulses. Indeed, a larger family of rhythms is obtained by considering all rotations of the timelines listed in Figures 12 and 14.

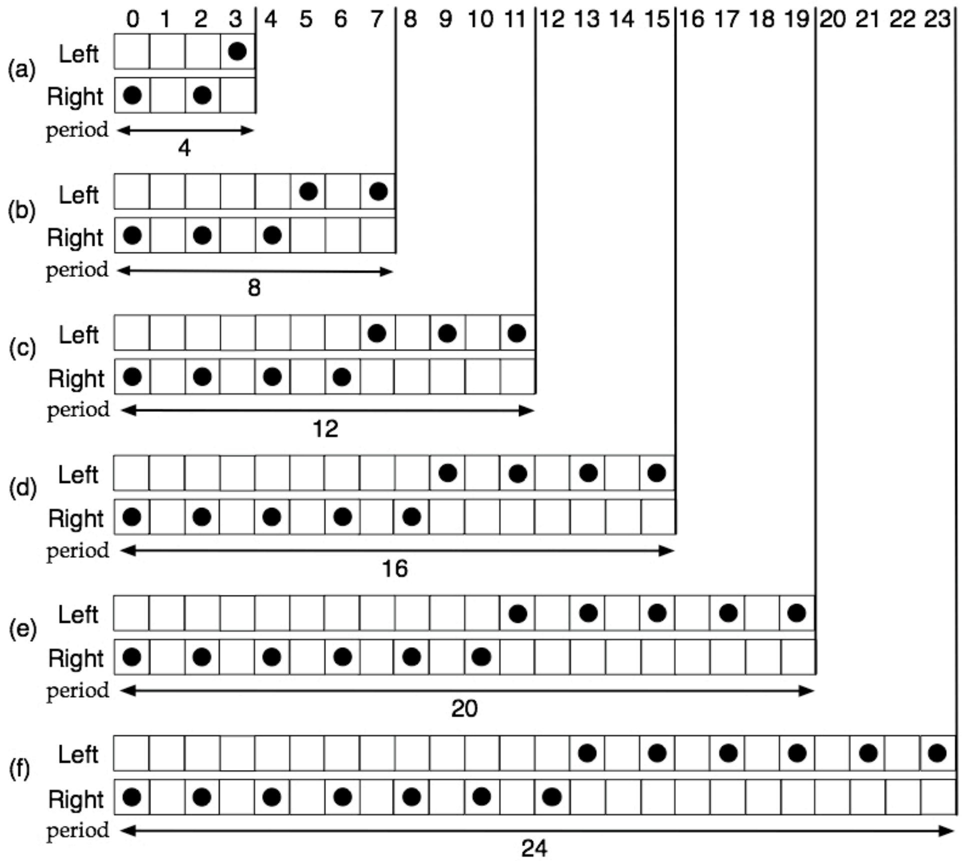

Fig. 14: A method for generating sharp toggle rhythms.

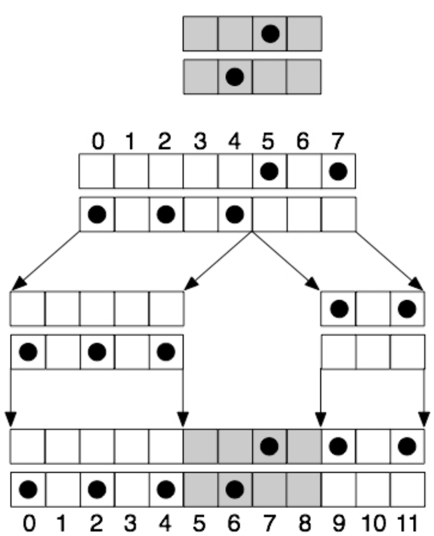

Fig. 15: Splicing a *sharp* toggle rhythm by inserting the right-left transition segment (shaded).

Since rhythm timelines are repeated throughout a piece, and are thus cyclic it is natural to represent toggle timelines using two concentric circles as pictured in Figure 16, where the outer and inner circles mark the right-hand and left-hand onsets, respectively, of a rotation of the standard pattern of Figure 14 (c). Since the rhythm has seven onsets, seven different timelines may be obtained by starting the cycle at any of the seven onsets. Indeed all such rhythms are used as timelines in different parts of Sub-Saharan Africa [3], [20], [22], [37]. For a second example consider the smooth toggle rhythm of Figure 12 (c) with 7 onsets and 16 pulses. When started on the third onset it is a timeline played with the wooden clave sticks in a version of the Brazilian samba [44: p. 64].

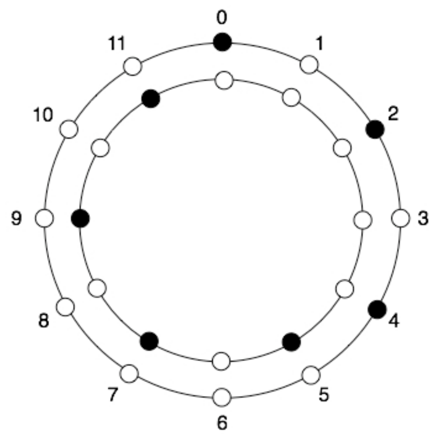

Fig. 16: A double-circle portrayal of a *toggle* rhythm: the right-hand and left-hand onsets are contained on the outer and inner circles, respectively, fusing together to yield the *standard pattern* [20].

The representation of cyclic rhythms on a circle permits an alternate definition of toggle rhythms based on the notion of linear separability in geometry. A rhythm (set of integer points on the circle) is a toggle rhythm if there exists a straight line that separates the left-hand onsets (on odd-numbered pulses) from the right-hand onsets (on even-numbered pulses). The rhythm in Figure 4, for example, is a toggle rhythm because there exists a line passing through two points: one being the midpoint between pulses 4 and 5, and another the midpoint between pulses 0 and 11, that leaves all the right-hand pulses on one side of this line, and the left-hand pulses on the other side. Note that although the two circles in Figure 4 are drawn as having different sizes for the sake of visualization, they should be considered as one and the same circle for this definition to remain valid.

Finally it should be noted that all the toggle rhythms considered heretofore have the property that the number of right-hand onsets differs by one from the number of left-hand onsets. This is not a requirement for a rhythm to belong to the toggle family. The clave son shown in circular toggle notation in Figure 17 (left) has only one of its five onsets played with the left hand, and yet it is a toggle rhythm since it admits a line that separates this onset from all the others, such as for example the line through pulses 1 and 5. On the other hand, the bossa-nova rhythm timeline of Figure 17 (right) with three right-hand onsets and two left-hand onsets is not a toggle rhythm since it does not admit any line that separates the onsets at pulses 3 and 13 from the other three.

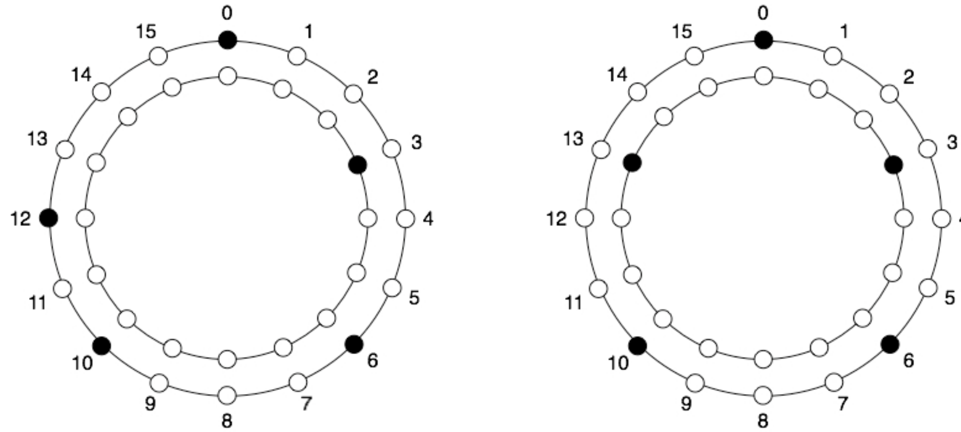

Fig. 17: The clave son (left) and the bossa-nova clave (right) in circular toggle notation.

4. Conclusion

If the right-hand and left-hand patterns of the sharp toggle rhythms of Figure 14 are combined into a single sequence then the timelines obtained bear a close resemblance to Kubik's *pyramid* of West and Central African asymmetric time-line patterns [24]. Kubik describes the structural relationships between some common African timeline patterns using the trapezoidal shape pictured in Figure 18. The pyramid is built from the top down starting with the 5-onset, 8-pulse rhythm in (a), which is partitioned into two pieces of 5 and 3 pulses given by [x x . x .] and [x x .], respectively. The next rhythm is constructed by inserting the pattern [x .] at the leftmost end of both parts to obtain the rhythm in (b). This approach is continued to create the rhythms in (c), (d), and (e). Note that putting together the left and right hand patterns in Figure 14, the rhythms in (b) - (f) are rotations of the five rhythms listed in Kubik's pyramid structure of Figure 18. For example, the 7-onset, 12-pulse rhythm in Figure 18 (b) is the same as that in Figure 14 (c) when started on the 6th onset on pulse number 9.

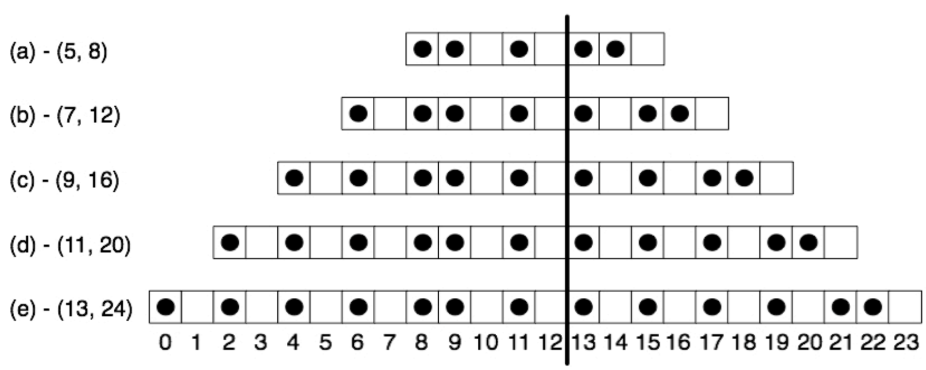

Fig. 18: Kubik's *pyramid* of West and Central African asymmetric time-line patterns [24].

The systematic methods presented here for generating and classifying good rhythm timelines within the broad context of designing tools for composition, and fitness functions for genetic algorithms, fall under the general umbrella that includes structural and generative methods for the analysis of the rhythm timelines of West and Central Africa in a cultural context advocated by ethnomusicologists and music theorists such as Agawu [20], [28], Anku [34]-[36], [43], Arom [18], [32]-[33], Kubik [24], Locke [40]-[41], Pressing [21], and Rahn [22]-[23]. It is hoped that the structural properties of the rhythm timelines explored here, their mathematical formulations, and the algorithms used to generate these rhythms, will help in the quest to determine a characterization of what makes a "good" rhythm good.

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