Constructing Convex 3-Polytopes from Two Triangulations of a Polygon

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Abstract

Guibas conjectured that given a convex polygon P in the xy-plane along with two triangulations of it, T_1 and T_2 that share no diagonals, it is always possible to assign height values to the vertices of P such that $P \cup T_1 \cup T_2$ becomes a convex 3-polytope. Dekster found a counter example but left open the questions of deciding if a given configuration corresponds to a convex 3-polytope, and constructing such realizations when they exist. This paper gives a proof that a relaxed version of Guibas' conjecture always holds true. The question of deciding the realizability of Guibas' conjecture is characterized in terms of a linear programming problem. This leads to an algorithm for deciding and constructing such realizations that incorporates a linear programming step with $O(n^2)$ inequality constraints and n variables.

1 Introduction

At the *First Canadian Conference on Computational Geometry* held at McGill University in August 1989 Leo Guibas conjectured that given a convex polygon P in the xy-plane along with two distinct (share no diagonals) triangulations of it, T_1 and T_2, it is always possible to perturb the vertices of P vertically out of the xy-plane to create a spatial polygon P' that projects to P so that the convex hull of P' becomes a convex 3-polytope whose skeleton projects onto $P \cup T_1 \cup T_2$ [5].

In 1995 Boris Dekster disproved Guibas' conjecture by showing that a necessary condition on the configuration P, T_1, T_2 could fail [3]. However, the problems of deciding when a given configuration corresponds to a convex 3-polytope in general, and constructing such realizations where possible were left open.

In this paper we introduce a new formalism for Guibas' conjecture based on generalized realizability problems. In this light, Guibas' conjecture is shown to be related to a variety of other problems. We then present one such problem that highlights the relationship between Guibas' conjecture and Steinitz's Theorem [7]. We propose a solution to the problem of deciding when a given configuration P, T_1, T_2 is realizable under the constraints present in Guibas' conjecture. Lastly we present an algorithm for computing realizations where possible, and find that the characterization of the decidability problem admits an immediate solution to the construction problem using linear programming with $O(n^2)$ inequality constraints and n variables.

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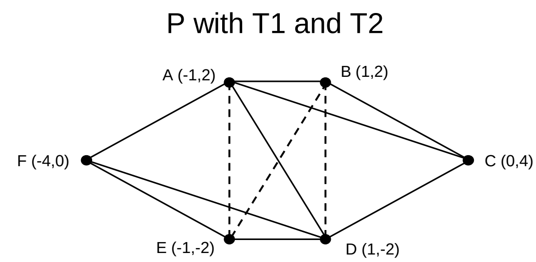

Figure 1: Dekster's counter example to Guibas' conjecture.

2 Dekster's Counter Example

Before presenting the new results we review Dekster's counter example to Guibas' conjecture. Dekster proves a general theorem outlining a necessary condition on the configuration P, T_1, T_2 in [3] using an elaborate geometric construction. He then presents an example where these properties do not hold implying that Guibas' conjecture is false. In this section a simplified version of Dekster's proof is sketched for the configuration used in his example. The configuration in question is pictured in Figure 1. This configuration cannot be realized as a convex 3-polytope under any assignment of height values to its vertices. Such a configuration is said to be *unrealizable*.

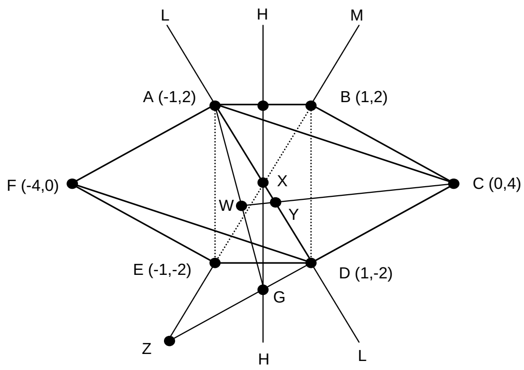

Figure 2: Dekster's Construction

Theorem 1 *The configuration (P, T_1, T_2) shown in Figure 1 does not have a realization as a convex 3-polytope.*

Lower case letters will denote points and lines in 3-space. Capital letters will denote the projection onto the xy-plane of points and lines in 3-space. The notation $[ijk]$ will be used to denote the plane of three vertices i, j, k.

Let L be the line through AD, and M be the line through BE. Now, construct a ray starting at C and passing though D. It intersects M at a point Z. Denote by X the point of intersection of AD and BE as seen in Figure 2.

Denote by h the intersection of the two planes $[afe]$ and $[bcd]$. Since these planes contain two faces of the bottom and are hinged along the bottom edges ae and bd that project onto parallel lines, the projection of the line h onto the plane forms a line H that must intersect both the segment AB and the segment ED. The line H is pictured here with an arbitrary placement. Since H crosses AB and ED it crosses the interior of the quadrilateral $ABDE$ thus it must also cross BE and DZ. Denote by G the point of intersection of H with DZ and note that whatever the exact placement of H the inequality $|DG| < |DZ|$ will always hold.

Since G is between Z and D it is always possible to find a point W that lies on the segment AG and is in the interior of the quadrilateral $AXEF$. The segment WC must then intersect the diagonal AD at a point Y that is inside the polygon (see Figure 2). Denote by w_1 the point contained in the plane $[aef]$ having projection W. Since $[aef]$ is the plane of a face of the bottom of P^* and W is in the interior of P it must be that w_1 is on or below the bottom of P^*. Let w_2 be any point in the interior of P^* also having projection W, then w_2 is above w_1. Also consider the point y_1 on the segment w_1c and the point y_2 on the segment w_2c both having projection Y. Since w_2 is above w_1 it must be that y_2 is above y_1.

Finally we derive a contradiction as follows: Let g be the point of $[aef]$ having projection G. By the above construction g lies in $[bcd]$ as well since g is on the line h, which is the intersection of the planes $[aef]$ and $[bcd]$ by definition. Then d lies on the segment gc. Since w_1 lies on ga by definition and d lies on gc we have that w_1c, and ad cross at y_1. y_1 is then an element of the top of P^*, so y_2 can not be above y_1 since y_1 is in the interior of P^*. This contradicts the above assertion that y_2 is above y_1 and thus the configuration (P, T_1, T_2) has no realization as a convex 3-polytope.

Dekster's proof relies on a complex geometric argument whereas the methods proposed in this paper focus on exploiting the strong connections between the geometry of 3-polytopes and graph theory.

3 General Realizability

A general realizability problem asks whether a given graph theoretic or geometric object ϕ can be embedded in a space σ under an isomorphic mapping f such that the mapped objects belong to the class ϕ'. A well know example is the problem of embedding a combinatorial graph $G = (V, E)$ in $\mathbb{R}^2$ as a planar graph $G' = (V', E')$. In this case the map m takes $v_i \in V$ to $v'_i = (x, y) \in V'$ where $(x, y) \in \mathbb{R}^2$, and m takes $e = (v_i, v_j) \in E$ to a curve $e' \in E'$ in $\mathbb{R}^2$ having endpoints v'_i and v'_j.

We will refer to a configuration consisting of a convex polygon $P = (V, E)$ with $v_i = (x, y) \in \mathbb{R}^2$, and two triangulations of it T_1 and T_2 by the triple (P, T_1, T_2). The underlying combinatorial graph structure will be denoted by $G = g(P, T_1, T_2)$. According to Guibas' conjecture the objects we are interested in mapping to are convex 3-polytopes $Q = (V', E')$ where $v'_i = (x', y', z') \in \mathbb{R}^3$. The isomorphic mapping m_g is such that each vertex $v_i = (x, y) \in V$ maps to a corresponding vertex $v'_i = (x, y, z') \in V'$, and each edge $e_k = (v_i, v_j) \in \{E \cup T_1 \cup T_2\}$ maps to an edge $e'_k = (v'_i, v'_j) \in E'$. Note that we do not allow the existing x and y coordinates of v_i to change under this mapping. This corresponds to simply raising the vertices of P vertically out of the plane to obtain Q as required by Guibas' conjecture. For related versions of realizability see [1], [2], [8].

4 A Relaxed Version of Guibas' Conjecture

In this section we show one relation between Guibas' conjecture and Steinitz's Theorem by introducing an intermediate problem that can be seen both as a relaxation of the conditions in Guibas' conjecture, and a special case of Steinitz's theorem.

Steinitz's Theorem: *A graph G is isomorphic to the edge graph of a convex 3-polytope Q if and only if G is 3-connected and planar.*

In this new problem the goal remains mapping configurations (P, T_1, T_2) to convex 3-polytopes $Q(V, E)$, but we relax the constraints on the mapping as follows: $m_r : v_i = (x, y) \in V \rightarrow v_i' = (x', y', z') \in V'$. In other words, we allow the position of $v_i \in V$ to be completely reassigned by the mapping m_r.

Theorem 2 *Given a convex polygon P in the plane, along with two distinct triangulations of it, T_1 and T_2, there always exists a mapping of the type m_r from (P, T_1, T_2) to a convex polytope $Q = (V', E')$ where $m_r : v_i = (x, y) \in V \rightarrow v_i' = (x', y', z') \in V'$, and edges are mapped in the natural way.*

Lemma 1 *The combinatorial graph G of (P, T_1, T_2) is planar.*

Since T_1 and T_2 are triangulations of P, and P is convex, we know that $G_1 = P \cup T_1$ and $G_2 = P \cup T_2$ are both plane graphs. To show that G is a planar graph we need only show that there exists a plane graph where the edges of either T_1 or T_2 are located on the exterior of P and do not cross. To construct such an embedding we go through the intermediate step of embedding G on the surface of the sphere. The vertices of P are mapped onto the equator of the sphere and the edges of P are mapped to arcs between the points along the equator. Since both T_1 and T_2 are sets of nonintersecting edges in the plane, they can each be mapped into a separate hemisphere as nonintersecting arcs of great circles. To recover a non-crossing plane embedding of G we first rotate the spherical embedding if necessary to ensure that the north pole does not lie on an embedded vertex or edge, and then take a stereographic projection onto the xy-plane. An illustration of such a transformation is shown in Figure 3.

Lemma 2 *The combinatorial graph G of (P, T_1, T_2) is 3-connected.*

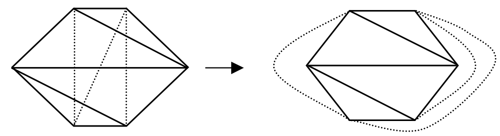

Figure 3: Transformation of $P \cup T_1 \cup T_2$ to a plane graph.

To see that Lemma 2 is true consider that each combinatorial graph G is the result of triangulating a convex polygon in two distinct ways. Each polygon triangulation has $n - 3$ diagonals for a total of $2n - 6$ diagonals from both triangulations. There are also n edges from the polygon itself. This gives a total of $3n - 6$ edges. A theorem of Whitney states that all planar graphs with $3n - 6$ edges are maximally planar. Lastly, a corollary to a theorem by Kuratowski states that all maximal planar graphs with $n \geq 4$ are 3-connected [4]. Thus the combinatorial graph G of any configuration (P, T_1, T_2) must be 3-connected.

The proof of Theorem 2 now follows from Lemmas 1, 2 and Steinitz' theorem.

A direct and elementary proof of Lemma 2 may also be obtained as follows. To show that G is 3-connected assume the contrary and derive a contradiction. Assume G is not 3-connected. Then there must exist some set of two vertices which disconnect the given graph. Without loss of generality we may assume that the two vertices in this set are v_0 and v_k where $k \neq 0, 1, n$. Since all the vertices in the graph lie on one large cycle induced by the edges of the polygon P, it should be intuitively obvious that the removal of two of these vertices can partition the graph into at most two connected components $A = \{v_1, v_2, ..., v_{k-1}\}$ and $B = \{v_{k+1}, v_{k+2}, ..., v_n\}$. By assumption the removal of v_0 and v_k disconnects the graph so it must be that there are no edges linking the vertices in A to the vertices in B. Thus A must be completely triangulated twice using only edges from $A \cup \{v_0, v_k\}$. However, any complete triangulation of a convex set of points must include all the edges in the convex hull of the point set. Thus we get that any triangulation of $A \cup \{v_0, v_k\}$ must have $v_0 v_k$ as an edge. Since we require that $A \cup \{v_0, v_k\}$ be triangulated twice, it follows that the edge $v_0 v_k$ must be in both T_1 and T_2. This is a contradiction since by assumption T_1 and T_2 are distinct sets. So there does not exist a set of two vertices which disconnects the graph, and thus G is 3-connected as required.

5 Deciding Realizability for Guibas' Conjecture

The method used to solve the decidability problem for Guibas' conjecture is inspired in part by a quantitative treatment of Steinitz's theorem due to Onn and Sturmfels [6]. Let $f_i = \{v_{i,1}, v_{i,2}, v_{i,3}\}$ be any face of the two dimensional plane embedding of (P, T_1, T_2), and let the vertices of f_i be given in counterclockwise order if $f_i \in T_1$, and in clockwise order if $f_i \in T_2$. One characterization of a convex 3-polytope is as the intersection of a set of half-spaces in 3-space. It follows that if (P, T_1, T_2) is realizable as a convex 3-polytope $Q = (V', E')$, all the vertices v_j in Q/f_i are on the same side of f_i, or Q cannot be the intersection of a set of half-spaces. This characterization yields a simple test for deciding realizability.

For each face $f_i = \{v_{i,1}, v_{i,2}, v_{i,3}\}$ with vertices given in the above order we require that every other vertex v_j of Q be on the same side of f_i. This condition can be checked by computing the volume of the 3-simplex $s_{i,j}$ formed by f_i with each v_j in Q/f_i. Since the faces of T_1 are given in clockwise order, and the faces of T_2 are given in counter clockwise order, the normals to the faces of Q will all point outwards. Thus we require that the volume of each 3-simplex $s_{i,j}$ be strictly positive. This indicates that for each face f_i all the other vertices of Q are on the same side of f_i. The volume of a 3-simplex can be easily computed using the determinant of the coordinate matrix as shown below. We want the volume of each $s_{i,j}$ to be strictly greater than 0 so we impose the condition shown in equation 1.

$$(1/6)\det\begin{pmatrix} x_{i,1} & y_{i,1} & z_{i,1} & 1 \\ x_{i,2} & y_{i,2} & z_{i,2} & 1 \\ x_{i,3} & y_{i,3} & z_{i,3} & 1 \\ x_j & y_j & z_j & 1 \end{pmatrix} > 0 \tag{1}$$

The key observation is that in the case of the present decision problem, the values of the x_j's and y_j's are specified by P in the given configuration (P, T_1, T_2). Thus the cofactor expansion of the z column in the above determinant yields an inequality that is linear in terms of the z coordinates as seen in equation 2.

$$z_{i,1}\det\begin{pmatrix} x_{i,2} & y_{i,2} & 1 \\ x_{i,3} & y_{i,3} & 1 \\ x_j & y_j & 1 \end{pmatrix} - z_{i,2}\det\begin{pmatrix} x_{i,1} & y_{i,1} & 1 \\ x_{i,3} & y_{i,3} & 1 \\ x_j & y_j & 1 \end{pmatrix}$$
$$+z_{i,3}\det\begin{pmatrix} x_{i,1} & y_{i,1} & 1 \\ x_{i,2} & y_{i,2} & 1 \\ x_j & y_j & 1 \end{pmatrix} - z_j\det\begin{pmatrix} x_{i,1} & y_{i,1} & 1 \\ x_{i,2} & y_{i,2} & 1 \\ x_{i,3} & y_{i,3} & 1 \end{pmatrix} > 0 \tag{2}$$

The simple manipulations above lead us directly to a solution to the problem of deciding when a configuration (P, T_1, T_2) is realizable as stated formally in Theorem 2.

Theorem 3 *A configuration (P, T_1, T_2) is realizable as a convex 3-polytope if and only if the there is a solution to the set S of linear inequalities given by:*

$$S = \bigcup_F \bigcup_V \left\{ z_{i,1}\det\begin{pmatrix} x_{i,2} & y_{i,2} & 1 \\ x_{i,3} & y_{i,3} & 1 \\ x_j & y_j & 1 \end{pmatrix} - z_{i,2}\det\begin{pmatrix} x_{i,1} & y_{i,1} & 1 \\ x_{i,3} & y_{i,3} & 1 \\ x_j & y_j & 1 \end{pmatrix} \right.$$
$$\left. +z_{i,3}\det\begin{pmatrix} x_{i,1} & y_{i,1} & 1 \\ x_{i,2} & y_{i,2} & 1 \\ x_j & y_j & 1 \end{pmatrix} - z_j\det\begin{pmatrix} x_{i,1} & y_{i,1} & 1 \\ x_{i,2} & y_{i,2} & 1 \\ x_{i,3} & y_{i,3} & 1 \end{pmatrix} > 0 \right\} \tag{3}$$

where F is the set of faces of G, V is the set of vertices of P, $(v_{i,1}, v_{i,2}, v_{i,3}) = f_i \in F$, and $v_j \in G/f_i$

If the set of inequalities S has a solution, then by construction when the value of each computed z_j is assigned to the z component of the corresponding $v_j \in P$, the resulting 3-polytope will be convex. On the other hand, if the system of inequalities has no solution, then for all possible assignments to the z_j's, at least one of the inequalities in S cannot be satisfied. If this is the case then there must exist four vertices where three are from the same face f_i, and one is from G/f_i such that the corresponding 3-simplex does not have strictly positive volume. Thus a convex 3-polytope cannot be formed for any possible assignments to the z_j's, and the configuration is not realizable. □

Since the realizability of a configuration (P, T_1, T_2) now depends only on the existence of a solution to the set of inequalities given in Theorem 2, a simple algorithm can be obtained for deciding realizability as outlined below. The algorithm uses linear programming to compute a solution.

Realization Decidability Algorithm:

1. Given a configuration (P, T_1, T_2) compute the combinatorial graph $G = g(P, T_1, T_2)$.
2. Compute the set of faces F of G generated by T_1 and T_2 in counterclockwise and clockwise order, respectively.
3. Apply Theorem 2 to compute the set of inequalities S.
4. Apply linear programming techniques to determine if the system of inequalities S has a solution.
5. If S has a feasible solution then (P, T_1, T_2) is realizable, otherwise (P, T_1, T_2) is not realizable.

As an example consider the configuration given by the polygon $P = (V, E)$ with $V = \{v_1 = (1,0), v_2 = (0,1), v_3 = (-1,0), v_4 = (0,-1)\}$, and $E = \{(v_1,v_2),(v_2,v_3),(v_3,v_4),(v_4,v_1)\}$ along with the triangulations $T_1 = \{(v_2,v_4)\}$ and $T_2 = \{(v_1,v_3)\}$. The set of faces given in the order explained above is $F = \{(v_1,v_3,v_2),(v_4,v_3,v_1),(v_1,v_2,v_4),(v_3,v_4,v_2)\}$. This configuration generates four identical inequalities when Theorem 2 is applied due to symmetry. The set of unique inequalities is $\{2z_1 - 2z_2 + 2z_3 - 2z_4 > 0\}$. This set of inequalities is clearly satisfiable by an assignment $z_i = 1$ for any i, and thus the configuration is realizable as a convex 3-polytope.

6 Constructing Realizations for Guibas' Conjecture

As it happens, the solution given in the previous section for deciding realizability under Guibas' conjecture immediately yields a solution to the problem of constructing realizations where possible. An algorithm for finding a realization if one exists is given below.

Realization Construction Algorithm:

1. Given a configuration (P, T_1, T_2) compute the combinatorial graph $G = g(P, T_1, T_2)$.
2. Compute the set of faces F of G generated by T_1 and T_2 in counterclockwise and clockwise order, respectively.
3. Compute the set of inequalities S, as explained in Theorem 2.
4. Apply linear programming techniques to determine if a feasible solution to the system of inequalities S exists.
5. If S has a feasible solution let it be $Z = (z_1, z_2, ..., z_n)$. Otherwise return null.
6. For each $v_j = (x_j, y_j) \in V$ construct $v'_j = (x_j, y_j, z_j) \in V'$ where z_j is given by Z.
7. For each $(v_i, v_j) \in \{E, T_1, T_2\}$ let $(v'_i, v'_j) \in E'$.
8. Return the 3-Polytope $Q = (V', E')$.

A slight variant this algorithm (implemented using the Maple V mathematical programming language) was used to check the configuration given by Dekster as a counter example to Guibas' conjecture. This configuration is pictured in Figure 1. The exact inputs passed to the implementation were as follows: $V = \{v_1 = (-1,2,0), v_2 = (1,2,1), v_3 = (4,0,1), v_4 = (1,-2,1), v_5 = (-1,-2,1), v_6 = (-4,0,-1)\}$. $F = \{(v_1,v_2,v_5),(v_2,v_4,v_5),(v_2,v_3,v_4),(v_5,v_6,v_1),(v_1,v_3,v_2),(v_1,v_4,v_3),(v_1,v_6,v_4),(v_6,v_5,v_4)\}$. The set of bounding (non-redundant) inequalities S_C induced by the configuration is shown below.

$$S_C = \left\{ \begin{array}{rl} 16z_4 - 16z_6 + 16z_1 - 16z_3 & > 0 \\ 20z_3 + 12z_6 - 16z_4 - 16z_2 & > 0 \\ -8z_5 + 8z_4 + 8z_1 - 8z_2 & > 0 \\ -4z_5 + 8z_3 - 20z_2 + 16z_1 & > 0 \\ 12z_5 + 8z_3 - 4z_2 - 16z_4 & > 0 \\ 12z_4 + 8z_6 - 4z_1 - 16z_5 & > 0 \\ 12z_3 + 20z_6 - 16z_5 - 16z_1 & > 0 \\ -4z_5 + 8z_6 + 12z_2 - 16z_1 & > 0 \\ -20z_5 + 8z_6 - 4z_2 + 16z_4 & > 0 \\ 4z_3 - 4z_6 - 16z_2 + 16z_1 & > 0 \\ 12z_1 + 8z_3 - 4z_4 - 16z_2 & > 0 \\ -16z_5 + 16z_4 - 4z_3 + 4z_6 & > 0 \\ -8z_6 - 16z_2 + 4z_4 + 20z_1 & > 0 \\ 4z_1 - 8z_3 + 20z_4 - 16z_5 & > 0 \end{array} \right\}$$

As expected, no solution was found for the set of inequalities S_C, thus providing a computer confirmation of the correctness of Dekster's counter example.

7 Computational Complexity

Although the dimension of the input *configuration* (P, T_1, T_2) is fixed at two, the corresponding linear programming *problem* has dimension equal to the number of vertices in (P, T_1, T_2). From the previous sections it follows that a configuration with n vertices is transformed into a linear programming problem having at most $n(n-3)$ inequalities. Thus the algorithms presented in the previous section are dominated by the linear programming step with $O(n^2)$ inequality constraints and n variables. There exists a variety of algorithms for linear programming and in practice at present there is rough parity between the simplex and interior point methods (see the enlightening survey by Todd [9]).

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