Interlocking Rhythms, Duration Interval Content, Cyclotomic Sets, and the Hexachordal Theorem

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We consider a rhythm to be represented by a subset of \( k \) points (onsets) on the circular lattice consisting of \( n \) points equally spaced on a circle [1]. Such sets are called cyclotomic sets in the crystallography literature [3], [13]. Every pair of such points determines an inter-onset-duration-interval (the geodesic distance between the pair of points on the circle) [14]. The histogram of this multiset of distances in the context of musical scales and chords is called its interval content [2]. We will use the same terminology for the case of rhythms, where the intervals are durations of time. The remaining \((n-k)\) lattice points determine a complementary rhythm [6]. A rhythm and its complement may thus be viewed as a pair of interlocking rhythms. Two rhythms which are not congruent but possess the same multiset of distances are said to be homometric, a term introduced by Lindo Patterson in 1939 [4].

First we review some of the history of interlocking rhythms in African and European music. Then we review the history of the hexachordal theorem in music theory [8], [9], [10], [11]. The hexachordal theorem states that two non-congruent interlocking rhythms with \( k = n/2 \) are homometric. The earliest proof of this theorem in the music literature appears to be due to Milton Babbitt and David Lewin [15], [8], [9], [10]. It used heavy machinery from topology. Later Lewin obtained new proofs using group theory. Later still Eric Regener [5] found an elementary simple proof of this theorem. The music theorists appear to be unaware that this theorem was known to crystallographers at least twenty years earlier [3]. It seems to have been proved by Lindo Patterson [3] around 1940 but he did not publish a proof. In the crystallography literature the theorem is called Patterson’s theorem [12]. The first published proof in the crystallography literature is due to Buerger [12]; it is based on image algebra, and is non-intuitive. A much simpler and elegant elementary proof was later found by Iglesias [13]. The simplest elementary proof was published by Steven Blau in 1999 [7]. We elucidate the proofs by Iglesias [13] and Blau [7].


