Enumeration of non-isomorphic canons

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Abstract

In this note we describe how to apply group actions in order to enumerate the isomorphism classes of rhythmic canons.

The notion of a rhythmic canon was originally formalized by O. Messiaen. He proposed to study canonical musical forms in which the imitation between the different voices concerns only rhythmical aspects, independent from melody or harmony. The first mathematical investigation of rhythmic canons goes back to Vuza's papers [7, 9, 8, 10] in which he actually introduced even much more complicated families of canons, the "Regular Complementary Canons of Maximal Category" (in short RCMC-canons). Their definition will be given below.

Before describing rhythmic canons in cyclic time, we view them as they may appear within a musical composition, in free linear time which has no cyclicity. Like in a melodic canon, one has several voices that may enter one after the other until all voices are present. As in the case of a melodic canon, all voices are just copies of a ground voice that is suitably translated in the time axis. For simplicity we suppose here that all voices are extended in both directions of the time axis ad infinitum. We further suppose that the ground voice is a periodic rhythm that we will call the *inner rhythm*. Following Vuza's definition, a *periodic rhythm* is an infinite subset R of the rationals $\mathbb{Q}$ (marking the attack times, or onsets) with $R = R + d$ for a suitable period d. Furthermore, R is supposed to be locally finite (i.e. the intersection of R with every time segment $[a, b]$ (of finite length $b - a \geq 0$) is finite). The *period* of a periodic rhythm R is the smallest positive rational number $d = d(R)$ satisfying $R = R + d$. We also mention another important characteristic of a periodic rhythm, its *pulsation* $p = p(R)$. It is defined as the $\max\{q \in \mathbb{Q}_+ \mid \forall r \in R - R\ \exists z \in \mathbb{Z}:\ r = zq\}$, in other words, it is the biggest rational q such that all distances between the attack points of R are integer multiples of q. Obviously, the period d is an integer multiple of the pulsation p.

Let A be the set of all rational numbers a such that the attack times of the voices of the canon can be expressed as $R + a$. Then A itself is a periodic rhythm, called the *outer rhythm* of the canon, with period $d(A)$, where $d(R)$ is an integer multiple of $d(A)$. Note that R and A may have different pulsations $p(R)$ and $p(A)$. Hence, the pulsation of a canon is defined as the greatest rational number p such that both $p(R)$ and $p(A)$ are integer multiples of p.

In order to switch from linear time to circular time the fraction $n = d(R)/p$ must be computed. Let r denote a fixed attack time within the inner rhythm R. Then each attack time in any of the voices is of the form $r + tp$ for a suitable integer t, i.e. the whole canon is contained in $r + p\mathbb{Z} \subset \mathbb{Q}$. Because everything is periodic with period $d(R)$, we can restrict to the factor space $(r + p\mathbb{Z})/d(R)\mathbb{Z}$ which may be identified with $\mathbb{Z}/n\mathbb{Z}$, the residue class ring of $\mathbb{Z}$ modulo $n\mathbb{Z}$.

From now on, we consider the whole canon within $Z_n := \mathbb{Z}/n\mathbb{Z}$ and assume that R, A, and V_a denote the projections of the inner rhythm, the outer rhythm, and the voices of a canon.

The concept of a canon used in the present paper is described by G. Mazzola in [4] and was presented by him to the author in the following way: A *canon* is a subset $K \subseteq Z_n$ together with a covering of K by pairwise different subsets, the voices, $V_i \neq \emptyset$ for $1 \leq i \leq t$, where $t \geq 1$ is the number of voices of the canon,

$$K = \bigcup_{i=1}^{t} V_i,$$

such that for all $i, j \in \{1, \ldots, t\}$

1. the set V_i can be obtained from V_j by a translation of Z_n,
2. there is only the identity translation which maps V_i to V_i,
3. the set of differences in K generates Z_n, i.e. $\langle K - K \rangle := \langle k - l \mid k, l \in K \rangle = Z_n$.

We prefer to write a canon K as a set of its subsets V_i. Two canons $K = \{V_1, \ldots, V_t\}$ and $L = \{W_1, \ldots, W_s\}$ are called *isomorphic* if $s = t$ and if there exists a translation T of Z_n and a permutation π in the symmetric group S_t such that $T(V_i) = W_{\pi(i)}$ for $1 \leq i \leq t$. Then obviously $T(K) = L$.

The aim of this note is to determine the number of non-isomorphic canons for given n. First we present the definition of canons in more details and give a short description of group actions, which are the standard tool to describe isomorphism classes of different objects (cf.[2, 3]). Since we are interested in the attack times modulo n, we assume that K is a subset of Z_n. The set of translations of Z_n is the cyclic group C_n generated by the permutation $\sigma_n := (0, 1, \ldots, n-1)$ which maps i to $i+1$ mod n. C_n acts in a natural way both on Z_n and on the set of all subsets of Z_n:

$$\sigma_n(i) = i + 1, \quad i \in Z_n, \text{ and } \sigma_n(A) = \{\sigma_n(i) \mid i \in A\}, \quad A \subseteq Z_n.$$

As usual we identify a subset A of Z_n with its *characteristic function* $\chi_A : Z_n \to \{0, 1\}$ given by

$$\chi_A(i) = \begin{cases} 1 & \text{if } i \in A \\ 0 & \text{otherwise.} \end{cases}$$

The set of all functions from a set X to a set Y will be indicated as Y^X. For arbitrary $f \in \{0, 1\}^{Z_n}$ let $\bar{f}$ denote the subset $f^{-1}(\{1\})$ of Z_n, whence $f = \chi_{\bar{f}}$.

Sometimes it is convenient to write functions f from Z_n to $\{0, 1\}$ as vectors of the form $(f(0), f(1), \ldots, f(n-1))$ using the natural order of the elements of Z_n, because then the set of all these functions is totally ordered by the *lexicographic order*. For functions $f, g \in \{0,1\}^{Z_n}$ we say $f < g$ if there exists an $i \in Z_n$ such that $f(j) = g(j)$ for $0 \leq j < i$ and $f(i) < g(i)$. The group action of C_n on the set of all subsets of Z_n described as an action on the set of all characteristic function is the following

$$C_n \times \{0,1\}^{Z_n} \to \{0,1\}^{Z_n} \qquad (\sigma_n^j, f) \mapsto f \circ \sigma_n^{-j}.$$

As the *canonical representative* of the *orbit* $C_n(f) = \{f \circ \sigma_n^j \mid 0 \leq j \leq n-1\}$ we choose the function $f_0 \in C_n(f)$ such that $f_0 \leq g$ for all $g \in C_n(f)$. Moreover, we choose $\bar{f}_0$ as the canonical representative of the orbit $C_n(\bar{f}) = \left\{\sigma_n^j(\bar{f}) \mid 0 \leq j \leq n-1\right\}$. In general, representatives of orbits $C_n(f)$ of functions f under the action of the cyclic group C_n are called *necklaces*.

Lemma 1. *If $f \neq 0$ denotes a function from Z_n to $\{0,1\}$, then the canonical representative f_0 of the orbit $C_n(f)$ fulfills $f_0(n-1) = 1$.*

A function $f \in \{0,1\}^{Z_n}$ (or the corresponding vector and the set $\bar{f}$) is called *acyclic* if $C_n(f)$ consists of n different objects. The canonical representative of the orbit of an acyclic function is usually called a *Lyndon word*. If f is acyclic, then all elements in the orbit $C_n(f)$ are acyclic as well, and we call $C_n(f)$ an *acyclic orbit*.

From the first two properties of a canon we derive that all voices V_i belong to the same orbit $C_n(V_1)$ and that each voice is acyclic. For that reason, we can describe a canon $K = \{V_1, \ldots, V_t\}$ as a pair (L, A) where L is the Lyndon word, which is the canonical representative of the orbit $C_n(\chi_{V_1})$, and $A = \{a_1, \ldots, a_t\}$ is a t-subset of Z_n such that $V_i = \sigma_n^{a_i}(L)$. In other words, L describes the inner rhythm and A the outer rhythm of K.

Now we analyse in which situations a pair (L, A), as described above, is not a canon, i.e. when it fails to fulfill the third property. Assume that (L, A) is not a canon, then the differences $K - K$ generate a proper subgroup of Z_n. This subgroup is isomorphic to $Z_{n/d}$ where $d > 1$ is a divisor of n. In other words, d is a divisor of all differences $k - l$ for all $k, l \in K$. We write $d \mid r$ in order to express that the integer d is a divisor of the integer r. (It is not connected with division in the ring Z_n.) It is easy to prove the following

Lemma 2. *Let $L \neq 0$ be a Lyndon word of length n, and let A be a t-subset of Z_n. The pair (L, A) does not describe a canon in Z_n if and only if there exists an integer $d > 1$ such that $d \mid n$, $d \mid k - l$ for all $k, l \in \bar{L}$, and $d \mid k - l$ for all $k, l \in A$.*

Consider the set S of all pairs (L, A) where L is a Lyndon word and A is a t-subset of Z_n. The group of translations of Z_n acts on S according to

$$C_n \times S \to S \qquad (\sigma_n^j, (L, A)) \mapsto (L, \sigma_n^j(A)).$$

If (L, A) is a canon, then the orbit $C_n(L, A) = (L, C_n(A))$ describes the isomorphism class of (L, A). In general, the canonical representative of $C_n(L, A)$ is (L, A_0) where A_0 is the canonical representative of $C_n(A)$. From Lemma 1 we already know that $\chi_{A_0}(n-1) = 1$ and $L(n-1) = 1$ if $L \neq 0$. This together with Lemma 2 proves

Lemma 3. *Let $L \neq 0$ be a Lyndon word of length n, and let A be a t-subset of Z_n. The pair (L, A) does not describe a canon in Z_n if and only if there exists a divisor $d > 1$ of n such that $L(i) = 1$ implies $i \equiv d - 1 \bmod d$, and $\chi_{A_0}(i) = 1$ implies $i \equiv d - 1 \bmod d$, where A_0 is the canonical representative of $C_n(A)$.*

In the next part we show that functions $f \in \{0,1\}^{Z_n}$ with the property $f(i) = 1$ implies $i \equiv -1 \bmod d$ (for $d \mid n$, $d > 1$) can be constructed from functions in $\{0,1\}^{Z_{n/d}}$. Let ψ_d be a function defined on $\{0,1\}$, such that $\psi_d(0)$ is the vector $(0, 0, \ldots, 0)$ consisting of d entries of 0, and $\psi_d(1) = (0, \ldots, 0, 1)$ is a vector consisting of $d - 1$ entries of 0 and 1 in the last position. We write the values of ψ_d in the form

$$\psi_d(0) = 0^d, \qquad \psi_d(1) = 0^{d-1}1.$$

If we apply ψ_d to each component of a vector $f \in \{0,1\}^{Z_r}$ by replacing each component 0 in f by 0^d and each component 1 by $0^{d-1}1$, we get a vector $\psi_d(f) \in \{0,1\}^{Z_{rd}}$ and

$$\psi_d(\{0,1\}^{Z_r}) = \left\{ f \in \{0,1\}^{Z_{rd}} \mid f(i) = 1 \text{ implies } i \equiv -1 \bmod d \right\}.$$

From Lemma 3 we conclude that (L, A) does not describe a canon if and only if there exists a $d > 1$ which divides n such that both L and χ_{A_0} are elements of $\psi_d(\{0,1\}^{Z_{n/d}})$.

Some properties of ψ_d are collected in the next

Lemma 4. *Let f, g be functions from Z_r to $\{0,1\}$. Then*

1. *$\psi_d(f) = \psi_d(g)$ if and only if $f = g$.*
2. *$\psi_d(f \circ \sigma_r) = \psi_d(f) \circ \sigma_{rd}^d$.*
3. *$\psi_d(C_r(f)) \subset C_{rd}(\psi_d(f))$.*
4. *$f < g$ if and only if $\psi_d(f) < \psi_d(g)$.*
5. *f_0 is the canonical representative of $C_r(f)$ if and only if $\psi_d(f_0)$ is the canonical representative of $C_{rd}(\psi_d(f))$.*
6. *ψ_d describes a bijection between the C_r-orbits on $\{0,1\}^{Z_r}$ and the C_{rd}-orbits on $\{0,1\}^{Z_{rd}}$ which have non-empty intersection with $\psi_d(\{0,1\}^{Z_r})$, thus*
$$|\{C_{rd}(A) \mid A \subseteq Z_{rd},\ A \neq \emptyset,\ d \mid k - l\ \forall k, l \in A\}| =$$
$$|\{C_r(A) \mid A \subseteq Z_r,\ A \neq \emptyset\}| =: \alpha(r).$$
7. *$f \neq 0$ is acyclic if and only if $\psi_d(f)$ is acyclic.*
8. *ψ_d describes a bijection between the acyclic C_r-orbits on $\{0,1\}^{Z_r}$ and the acyclic C_{rd}-orbits on $\{0,1\}^{Z_{rd}}$ which have non-empty intersection with $\psi_d(\{0,1\}^{Z_r})$, thus*
$$|\{C_{rd}(A) \mid A \subseteq Z_{rd},\ A \neq \emptyset,\ d \mid k - l\ \forall k, l \in A,\ A \text{ acyclic}\}| =$$
$$|\{C_r(A) \mid A \subseteq Z_r,\ A \neq \emptyset,\ A \text{ acyclic}\}| =: \lambda(r).$$

9. Both f and $\psi_d(f)$ have the same number of components which are 1.

Proof. Since ψ_d is an injective mapping from each component of f into the set $\{0^d, 0^{d-1}1\}$, the first statement is clear.

In order to show that the second item is true, assume that $\psi_d(f \circ \sigma_r)(j) = 1$ for some $j \in Z_{rd}$. According to the definition of ψ_d, this is equivalent to $j = sd - 1$ and $(f \circ \sigma_r)(s-1) = 1$ for some $s \in \{1, \ldots, r\}$. In other words, $j = sd - 1$ and $f(s) = 1$, which is equivalent to $j = sd - 1$ and $\psi_d(f)((s+1)d - 1) = 1$. This, however, is the same as $\psi_d(f)(j + d) = 1$, whence $(\psi_d(f) \circ \sigma_{rd}^d)(j) = 1$. If $f = 0$ the assertion is always true.

The third part is an immediate consequence of the second.

If $f < g$, then there exists $i \in Z_r$ such that $f(j) = g(j)$ for all $j < i$ and $f(i) < g(i)$, whence $f(i) = 0$ and $g(i) = 1$. For that reason, $\psi_d(f(i)) = 0^d$ and $\psi_d(g(i)) = 0^{d-1}1$. Hence, $\psi_d(f)(j) = \psi_d(g)(j)$ for $j < id-1$ and $\psi_d(f)(id-1) = 0 < 1 = \psi_d(g)(id-1)$, and consequently $\psi_d(f) < \psi_d(g)$. If, conversely, $\psi_d(f) < \psi_d(g)$, then there exists an $i \in Z_{rd}$ such that $\psi_d(f)(j) = \psi_d(g)(j)$ for all $j < i$ and $\psi_d(f)(i) < \psi_d(g)(i)$, whence $\psi_d(g)(i) = 1$ and $i \equiv -1 \bmod d$. Assume that i is of the form $sd - 1$. Then $\psi_d(f(j)) = \psi_d(g(j))$ for $j < s$, and $\psi_d(f(s)) = 0^d$ whereas $\psi_d(g(s)) = 0^{d-1}1$. Since ψ_d is an injective mapping, $f(j) = g(j)$ for $j < s$ and $f(s) = 0 < 1 = g(s)$, which implies that $f < g$.

From the definition of the canonical representative of an orbit and from the items 4. and 2. it follows immediately that $\psi_d(f_0) < \psi_d(f_0 \circ \sigma_r^j) = \psi_d(f_0) \circ \sigma_{rd}^{dj}$ for all $j \in \{1, \ldots, r-1\}$. Moreover, if $n \not\equiv 0 \bmod d$, then $(\psi_d(f_0) \circ \sigma_{rd}^n)(rd-1) = \psi_d(f_0)(n-1) \neq 1$, since $n-1 \not\equiv -1 \bmod d$. According to Lemma 2, the representative $\psi_d(f_0) \circ \sigma_{rd}^n$ cannot be the canonical representative of the orbit $C_{rd}(\psi_d(f_0))$. Hence, the canonical representative is $\psi_d(f_0)$.

Then 6. is a trivial consequence of 5. Similar arguments can be applied for proving 7. and 9. Item 8. follows immediately from 7. $\square$

In order to compute the number of all C_r-orbits on $\{0,1\}^{Z_r}$ we apply *Pólya's theory* cf. [2, 3, 5, 6]. In the present situation we have to compute the cycle index of the group C_r acting on Z_r which is a polynomial in $z_1, \ldots, z_r$ over $\mathbb{Q}$ given by

$$C(C_r, Z_r) = \frac{1}{r} \sum_{s|r} \varphi(s) z_s^{r/s},$$

where φ is the *Euler totient function.* Replacing each indeterminate z_i in $C(C_r, Z_r)$ by $|\{0,1\}|$, which is equal to 2, we compute the number of all C_r-orbits on $\{0,1\}^{Z_r}$ as

$$C(C_r, Z_r, z_i := 2) = \frac{1}{r} \sum_{s|r} \varphi(s) 2^{r/s}.$$

If we replace z_i by $1 + z^i$, where z is an indeterminate over $\mathbb{Q}$, then the number of C_r-orbits of k-sets $A \subseteq Z_r$ is the coefficient of z^k in

$$C(C_r, Z_r, z_i := 1 + z^i) = \frac{1}{r} \sum_{s|r} \varphi(s)(1 + z^i)^{r/s}.$$

In conclusion (for the definition of $\alpha(r)$ see 6. in Lemma 4)

$$\alpha(r) = C(C_r, Z_r, z_i := 2) - 1,$$

since we don't count the orbit of the empty set. Similar methods can be applied to enumerate the number of acyclic C_r-orbits on $\{0,1\}^{Z_r}$ or, what is equivalent to this, to enumerate all Lyndon words of length r over the alphabet $\{0,1\}$. For $r > 1$, this number is given as (for the definition of $\lambda(r)$ see 8. in Lemma 4)

$$\lambda(r) = \frac{1}{r}\sum_{s|r} \mu(s) 2^{r/s},$$

where μ is the *classical Moebius function*. Moreover, the number of Lyndon words of length r over $\{0,1\}$ having r_1 components 1 and $r - r_1$ components 0 is

$$\frac{1}{r}\sum_{s|\gcd(r-r_1,r_1)} \mu(s)\binom{r/s}{r_1/s}.$$

And the generating function of Lyndon words of length r over the alphabet $\{0,1\}$ with r_1 components 1 is of the form

$$\frac{1}{r}\sum_{r_1=1}^{r-1}\left(\sum_{s|\gcd(r-r_1,r_1)} \mu(s)\binom{r/s}{r_1/s}\right) y^{r_1}.$$

In the case $r = 1$ there are two Lyndon words, namely $f_0 = (0)$ and $f_1 = (1)$. The first one is the characteristic function of the empty set in Z_1 so we don't want to count it. For that reason, we must set $\lambda(1) = 1$. Moreover, it should be mentioned that f_0 is the unique Lyndon word such that $\psi_d(f_0)$ is not a Lyndon word for $d > 1$.

Finally, we have to combine all these results for enumerating the isomorphism classes of canons. Let $n \geq 1$ be an integer. For any divisor $d \geq 1$ of n, let $M_{n,d}$ be the set of all pairs $(L, C_n(A))$, where L is a Lyndon word of length n over $\{0,1\}$, (in the case $n = 1$ different from 0) such that $L(i) = 1$ implies $i \equiv -1 \bmod d$, and A is a non-empty subset of Z_n, such that $d \mid k - l$ for all $k, l \in A$. Hence

$$M_{n,d} = \left\{ (L, C_n(A)) \mid \begin{array}{l} L \text{ Lyndon word}, \ L \neq 0, \ L(i) = 1 \Rightarrow i \equiv -1 \bmod d \\ A \subseteq Z_n, \ A \neq \emptyset, \ d \mid k - l \ \forall k, l \in A \end{array} \right\}.$$

From Lemma 4 we deduce that ψ_d describes a bijection between $M_{n,d}$ and $M_{n/d,1}$, thus

$$|M_{n,d}| = \left|M_{n/d,1}\right| = \lambda(n/d)\alpha(n/d),$$

which is the number of possibilities to choose L and $C_n(A)$ according to the desired properties of $M_{n,d}$. Finally, let κ_n be the set of isomorphism classes of canons in Z_n,

$$\kappa_n = \{(L, C_n(A)) \in M_{n,1} \mid (L, A) \text{ describes a canon}\}.$$

From Lemma 3 we deduce that

$$\kappa_n = M_{n,1} \setminus \bigcup_{\substack{d>1 \\ d|n}} M_{n,d}.$$

Theorem. *The number of isomorphism classes of canons in Z_n is*

$$|\kappa_n| = \sum_{d|n} \mu(d)\lambda(n/d)\alpha(n/d).$$

Proof. First we prove that the set $M_{n,1}$ is the disjoint union

$$M_{n,1} = \dot{\bigcup_{d|n}} \psi_d(\kappa_{n/d}),$$

where

$$\psi_d(\kappa_{n/d}) = \left\{\psi_d(L, C_{n/d}(A)) \mid (L, C_{n/d}(A)) \in \kappa_{n/d}\right\}$$

and

$$\psi_d(L, C_{n/d}(A)) = (\psi_d(L), C_n(\overline{\psi_d(\chi_A)})).$$

It is clear that $M_{n,1}$ contains this union. Moreover, this union is disjoint, since for a canon $K = (L, C_{n/d}(A)) \in \kappa_{n/d}$ we have

$$\langle K - K \rangle = Z_{n/d} \cong \langle \psi_d(K) - \psi_d(K) \rangle.$$

Finally, we have to show that each element of $M_{n,1}$ belongs to this union. If $(L, C_n(A)) \in M_{n,1}$, then choose the biggest d such that $(L, C_n(A))$ belongs to $M_{n,d}$. (I.e. $(L, C_n(A))$ does not belong to $M_{n,dd'}$ for $d' > 1$. It is always possible to find such d, since, if $(L, C_n(A))$ belongs both to M_{n,d_1} and M_{n,d_2}, then it also belongs to $M_{n,\mathrm{lcm}(d_1,d_2)}$.) Then there exists $(L', C_{n/d}(A')) \in M_{n/d,1}$ such that $L = \psi_d(L')$ and $\chi_{A_0} = \psi_d(\chi_{A'_0})$ for the canonical representatives A_0 and A'_0 of $C_n(A)$ and $C_{n/d}(A')$. Moreover, $(L', C_{n/d}(A')) \in \kappa_{n/d}$ because otherwise there would be some $d' > 1$ which is a divisor of n/d such that $(L', C_{n/d}(A')) \in M_{n/d,d'}$. But then $(L, C_n(A))$ also belongs to $M_{n,dd'}$, which is a contradiction to the choice of d.

Hence,

$$|M_{n,1}| = \sum_{d|n} \left|\kappa_{n/d}\right| = \sum_{d|n} |\kappa_d|,$$

and by Moebius inversion we get

$$|\kappa_n| = \sum_{d|n} \mu(n/d)\,|M_{d,1}| = \sum_{d|n} \mu(d)\left|M_{n/d,1}\right| = \sum_{d|n} \mu(d)\,|M_{n,d}| = \sum_{d|n} \mu(d)\lambda(n/d)\alpha(n/d),$$

which finishes the proof. $\square$

If λ and α are replaced by the corresponding generating functions

$$\bar{\lambda}(r) = \begin{cases} \frac{1}{r}\sum_{r_1=1}^{r-1}\left(\sum_{s|\gcd(r-r_1,r_1)}\mu(s)\binom{r/s}{r_1/s}\right)y^{r_1} & \text{if } r>1 \\ y & \text{if } r=1 \end{cases}$$

and

$$\bar{\alpha}(r) = C(C_r, Z_r, z_i := 1+z^i) - 1,$$

then the coefficient of $y^s z^t$ in

$$\sum_{d|n}\mu(d)\bar{\lambda}(n/d)\bar{\alpha}(n/d)$$

gives the number of non-isomorphic canons in Z_n which consist of t voices where each voice contains exactly s attack times.

For example the numbers of isomorphism classes of canons in Z_n for $1 \leq n \leq 10$ are:

n	1	2	3	4	5	6	7	8	9	10
$\lvert\kappa_n\rvert$	1	1	5	13	41	110	341	1035	3298	10550

Finally, we have a closer look at the 13 isomorphism classes of canons in Z_4, which can be classified according to their number of voices and attack times per voice by $z^4y^3 + z^4y^2 + z^4y + z^3y^3 + z^3y^2 + z^3y + 2z^2y^3 + 2z^2y^2 + z^2y + zy^3 + zy^2$. From this list we can read, for instance, that there are exactly three classes of canons with 4 voices which have 1, 2, or 3 attack times per voice. Or we see that there are five classes of canons with 2 voices, two of them having 3 attack times per voice, two of them having 2 attack times per voice, and only one having 1 attack time in each of its two voices.

We can even compute the canonical representative of each isomorphism class. For instance, there are exactly three Lyndon words L_1, L_2, L_3 of length 4 over $\{0,1\}$ and five representatives $f_1, \ldots, f_5$ of the C_4-orbits of non empty subsets of Z_4:

$$\begin{array}{ll} L_1 = (0,0,0,1) & f_1 = (0,0,0,1) \\ L_2 = (0,0,1,1) & f_2 = (0,0,1,1) \\ L_3 = (0,1,1,1) & f_3 = (0,1,0,1) \\ & f_4 = (0,1,1,1) \\ & f_5 = (1,1,1,1) \end{array}$$

The Lyndon words describe the distribution of the attack times in one voice (i.e. the inner rhythm of the canon), the necklaces describe the distribution of the voices over the complete period, here in this example over Z_4, which is the outer rhythm. Only L_1 can be constructed as $\psi_4((1))$ or $\psi_2((0,1))$ from shorter Lyndon words, so all pairs (L_i, f_j) for $i \in \{2,3\}$ and $j \in \{1,\ldots,5\}$ describe canons in Z_4. Since $f_1 = \psi_4((1)) = \psi_2((0,1))$ and $f_3 = \psi_2((1,1))$, and (1), $(0,1)$, and $(1,1)$ are necklaces of length 1 or 2 over $\{0,1\}$, the pairs $(L_1, C_4(f_1)) = \psi_4((1), C_1(1))$ and $(L_1, C_4(f_3)) = \psi_2((0,1), C_2(1,1))$ do not belong to κ_4. But the pairs (L_1, f_j) for $j \in \{2,4,5\}$ describe again canons in Z_4.

From this example we also see how to compute complete lists of isomorphism classes of canons in Z_n for arbitrary n. There exist fast algorithms for computing all Lyndon words and all necklaces of length n over $\{0,1\}$. Then each pair (L_i, f_j) must be tested whether there exists some $d > 1$ such that $(L_i, f_j) = (\psi_d(L'), \psi_d(f'))$ for a Lyndon word L' and a necklace f' of length n/d.

There exist more complicated definitions of canons. A pair (R, A) of inner and outer rhythm defines a *rhythmic tiling canon* with voices V_a for $a \in A$ in Z_n if and only if

1. the voices V_a cover entirely the cyclic group Z_n,

2. the voices V_a are pairwise disjoint.

It is obvious that periods and pulsations can be determined in cyclic time as well. Rhythmic tiling canons with the additional property,

3. the periods $d(R)$ and $d(A)$ coincide,

are called *regular complementary canons of maximal category.* So far we were not dealing with that kind of canons.

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