The *Euclidean* Algorithm Generates Traditional Musical Rhythms

Godfried Toussaint*
School of Computer Science
Centre for Interdisciplinary Research in Music Media and Technology
The Schulich School of Music
McGill University
Montréal, Québec, Canada
godfried@cs.mcgill.ca

Extended version of the paper that appeared in the *Proceedings of BRIDGES: Mathematical Connections in Art, Music and Science*, Banff, Alberta, Canada, July 31-August 3, 2005, pp. 47-56.

Abstract

The *Euclidean* algorithm (which comes down to us from Euclid's *Elements*) computes the greatest common divisor of two given integers. It is shown here that the structure of the Euclidean algorithm may be used to automatically generate, very efficiently, a large family of rhythms used as timelines (rhythmic ostinatos), in traditional world music. These rhythms, here dubbed *Euclidean* rhythms, have the property that their onset patterns are distributed as evenly as possible in a mathematically precise sense, and optimal manner. *Euclidean* rhythms are closely related to the family of *Aksak* rhythms studied by ethnomusicologists, and occur in a wide variety of other disciplines as well. For example they characterize algorithms for drawing digital straight lines in computer graphics, as well as algorithms for calculating leap years in calendar design. *Euclidean* rhythms also find application in nuclear physics accelerators and in computer science, and are closely related to several families of words and sequences of interest in the study of the combinatorics of words, such as *mechanical* words, *Sturmian* words, *two-distance* sequences, and *Euclidean strings*, to which the *Euclidean* rhythms are compared.

1. Introduction

What do African bell rhythms [126], spallation neutron source (SNS) accelerators in nuclear physics [18], Sturmian words and string theory (stringology) in computer science [85], Markoff numbers and *two-distance* sequences in number theory [114], [86], [28], drawing digital straight lines in computer graphics [72], calculating leap years in calendar design [61], [9], and an ancient algorithm [55] (called the *Euclidean Algorithm* in computer science) for computing the greatest common divisor of two numbers, originally described by Euclid [52], have in common? The short answer is: *patterns distributed as evenly as possible*. For the long answer please read on.

Mathematics and music have been intimately intertwined at least since the day Pythagoras discovered 2500 years ago that the pleasing experience of musical harmony is the result of ratios of small integers [10]. However, most of this interaction has been in the domains of pitch, scales, and tuning systems. For some historical snapshots of this interaction the reader is referred to H. S. M. Coxeter's delightful account [41]. Rhythm, on the other hand has been throughout history, until recently, mostly ignored. Some notable recent works on the subject are the books by Simha Arom [6], Christopher Hasty [63], and Justin London [84]. The earlier books by Grosvenor Cooper and Leonard B. Meyer [38] and Maury Yeston [138] are also useful.

*This research was supported by NSERC and FCAR.

In this paper we make some mathematical connections between musical rhythm and other areas of knowledge such as nuclear physics, calendar design, number theory, geometry, and computer science, as well as the work of another famous ancient Greek mathematician, Euclid of Alexandria. It should be noted that the Euclidean Algorithm has been connected to music theory previously by Viggo Brun [25]. However, Brun used Euclidean algorithms to calculate the lengths of strings in musical instruments in between two lengths l and $2l$, so that all pairs of adjacent strings have the same length-ratios. Here on the other hand the Euclidean algorithm is related to rhythm; it is shown that this algorithm generates almost all rhythmic timelines used in traditional world music.

During the past thirty years a number of researchers have approached the study of rhythmic timelines using *generative* methods, notably Kubik [77], Locke [80], Pressing [101], Rahn [105], [106], Anku [4], Toussaint [125], [126], [127], [128], and Agawu [1]. Agawu [1] provides an in-depth analysis of these methods applied to African timelines. On the other hand, the Euclidean algorithm exposed here is a mathematical model of rhythmic timeline generation that applies to music from all over the world (with the exception of India). It should be stressed that this is not a model of the conscious *process* by which musicians in any culture arrive at their preferred timelines, but rather of the *inherent properties* (both mathematical and musicological) of the resulting timelines obtained.

2. Timing Systems in Neutron Accelerators

The following problem is considered by Bjorklund [18], [17] in connection with the operation of certain components (such as high voltage power supplies) of spallation neutron source (SNS) accelerators used in nuclear physics. Time is divided into intervals (in the case of SNS, 10 seconds). During some of these intervals a gate is to be enabled by a timing system that generates pulses that accomplish this task. The problem for a given number n of time intervals, and another given number $k < n$ of pulses, is to distribute the pulses as evenly as possible among these n intervals. Bjorklund [18] represents this problem as a binary sequence of k ones and $n-k$ zeros, where each integer represents a time interval, and the ones represent the times at which the pulses occur. The problem then reduces to the following: construct a binary sequence of n bits with k ones, such that the k ones are distributed as evenly as possible among the $(n-k)$ zeros. If k divides evenly (without remainder) into n, then the solution is obvious. For example, if $n = 16$ and $k = 4$, the solution is [1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0]. The solution is less obvious when k and n are relatively prime numbers [100], i.e., when k and n are not evenly divisible by any number other than 1.

Bjorklund's algorithm will be described simply by using one of his examples. Consider a sequence with $n = 13$ and $k = 5$. Since $13 - 5 = 8$, we start by considering a sequence consisting of 5 ones followed by 8 zeros which should be thought of as 13 sequences of one bit each:

[1][1][1][1][1][0][0][0][0][0][0][0][0]

If there is more than one zero the algorithm moves zeros in stages. We begin by taking zeros one at-a-time (from right to left), placing a zero after each one (from left to right), to produce five sequences of two bits each, with three zeros remaining:

[10] [10] [10] [10] [10] [0] [0] [0]

Next we distribute the three remaining zeros in a similar manner, by placing a [0] sequence after each [10] sequence to obtain:

[100] [100] [100] [10] [10]

Now we have three sequences of three bits each, and a remainder of two sequences of two bits each. Therefore we continue in the same manner, by placing a [10] sequence after each [100] sequence to obtain:

[10010] [10010] [100]

The process stops when the remainder consists of only one sequence (in this case the sequence [100]), or we run out of zeros (there is no remainder). The final sequence is thus the concatenation of [10010], [10010], and [100]:

[1 0 0 1 0 1 0 0 1 0 1 0 0]

Note that one could proceed further in this process by inserting [100] into [10010] [10010]. However, Bjorklund argues that since the sequence is cyclic it does not matter (hence his stopping rule). For the same reason, if the initial sequence has a group of ones followed by only one zero, the zero is considered as a remainder consisting of one sequence of one bit, and hence nothing is done. Bjorklund [18] shows that the final sequence may be computed from the initial sequence using $O(n)$ arithmetic operations in the worst case.

A more convenient and visually appealing way to implement this algorithm by hand is to perform the sequence of insertions in a vertical manner as follows.

First take five zeros from the right and place them under the 5 ones on the left to obtain:

```
1 1 1 1 1 0 0 0
0 0 0 0 0
```

Then move the three remaining zeros in a similar manner to obtain:

```
1 1 1 1 1
0 0 0 0 0
0 0 0
```

Next place the two remainder columns on the right under the two leftmost columns to obtain:

```
1 1 1
0 0 0
0 0 0
1 1
0 0
```

Here the process stops because the remainder consists of only one column. The final sequence is obtained by concatenating the three columns from left to right to obtain:

1 0 0 1 0 1 0 0 1 0 1 0 0

3. The Euclidean Algorithm

One of the oldest well known algorithms, described in Euclid's *Elements* (*circa* 300 B.C.) in Proposition 2 of *Book VII*, today referred to as the *Euclidean algorithm*, computes the greatest common divisor of two given integers [52], [55]. Indeed, Donald Knuth [75] calls this algorithm the "granddaddy of all algorithms, because it is the oldest nontrivial algorithm that has survived to the present day." The idea is very simple. Repeatedly replace the larger of the two numbers by their difference until both are equal. This final number is then the greatest common divisor. Consider as an example the numbers 5 and 8 as before. First, 8 minus

5 equals 3; then 5 minus 3 equals 2; next 3 minus 2 equals 1; and finally 2 minus 1 equals 1. Therefore the greatest common divisor of 5 and 8 is 1, or in other words 5 and 8 are relatively prime numbers. The algorithm may also be described succinctly in a recursive manner as done in [40]. Let m and k be the input integers with $m > k$.

EUCLID(m, k)
1. **if** $k = 0$
2. **then return** m
3. **else return** EUCLID(k, $m \bmod k$)

Running this algorithm with $m = 8$ and $k = 5$ we obtain:
EUCLID(8,5) = EUCLID(5,3) = EUCLID(3,2) = EUCLID(2,1) = EUCLID(1,0) = 1

It is clear from the description of the Euclidean algorithm that if m and k are equal to the number of zeros and ones, respectively, in a binary sequence (with $n = m + k$) then Bjorklund's algorithm described in the preceding has the same structure as the Euclidean algorithm. Indeed, Bjorklund's algorithm uses the repeated subtraction form of division, just as Euclid did in his *Elements* [52]. It is also well known that if algorithm EUCLID(m, k) is applied to two $O(n)$ bit numbers (binary sequences of length n) it will perform $O(n)$ arithmetic operations in the worst case [40].

4. Euclidean Rhythms in Traditional World Music

A common method of representing musical rhythms is as binary sequences, where each bit (called a *pulse* in this context) is considered as one unit of time (for example a 16*th* note), a *zero* bit represents a silence (or a 16*th* rest), and a *one* bit represents an attack (or onset) of a note [125]. Therefore, the binary sequences generated by Bjorklund's algorithm, as described in the preceding, may be considered as one family of rhythms. Furthermore, since Bjorklund's algorithm is a way of visualizing the repeated-subtraction version of the Euclidean algorithm, these rhythms will be called *Euclidean rhythms*, and denoted by $E(k, n)$, where k denotes the number of ones (onsets) and n (the number of pulses) is the length of the sequence (zeros plus ones). For example $E(5, 13) = [1001010010100]$. The zero-one notation is not ideal for representing binary rhythms because it is difficult to visualize the locations of the onsets as well as the duration of the inter-onset intervals. In the musicology literature it is common to use the symbol 'x' for for the one bit and the symbol '.' for the zero bit. In this more iconic notation the preceding rhythm is written as E(5,13) = [x . . x . x . . x . x . .].

The rhythm E(5,13) is a cyclic rhythm with a time span (measure) of 13 units. Although it is used in Macedonian music [7], it is considered to be a relatively rare measure in world music. Let us consider for contrast two widely used values of k and n; in particular, what is $E(3, 8)$? Applying the Euclidean algorithm to the corresponding sequence [1 1 1 0 0 0 0 0], the reader may easily verify that the resulting Euclidean rhythm is E(3,8) = [x . . x . . x .]. This rhythm is illustrated in Figure 1 (a) as a triangle (polygon in general), yet another useful way to represent cyclic rhythms [125]. Here the rhythm is assumed to start at the location labelled 'zero', time flows in a clockwise direction, and the numbers by the sides of the triangle indicate the inter-onset duration intervals. Indeed, an even more compact representation of the rhythm is the adjacent-inter-onset-interval vector, namely (332), which will also be used here.

The Euclidean rhythm E(3,8) pictured in Figure 1 (a) is none other than one of the most famous on the planet. In Cuba it goes by the name of the *tresillo* and in the USA is often called the *Habanera* rhythm used in hundreds of *rockabilly* songs during the 1950's. It can often be heard in early rock-and-roll hits in the left-hand patterns of the piano, or played on the string bass or saxophone [22], [69], [96]. A good example

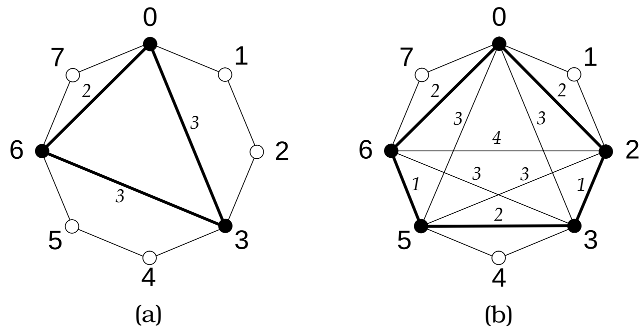

Figure 1: *(a) The Euclidean rhythm $E(3,8)$ is the Cuban* tresillo*, (b) The Euclidean rhythm E(5,8) is the Cuban* cinquillo*.*

is the bass rhythm in Elvis Presley's *Hound Dog*. The tresillo pattern is also found widely in West African traditional music. For example, it is played on the *atoke* bell in the *Sohu*, an *Ewe* dance from Ghana [70]. The tresillo can also be recognized as the first bar (8 pulses) of the ubiquitous two-bar *clave Son* given by [x . . x . . x . . . x . x . . .].

In the two examples considered in the preceding (E(5,13) and E(3,8)) the number of ones is less than the number of zeros. If instead the number of ones is greater than the number of zeros, Bjorklund's algorithm yields the following steps with, for example $k = 5$ and $n = 8$.

[1 1 1 1 1 0 0 0]
[10] [10] [10] [1] [1]
[101] [101] [10]
[1 0 1 1 0 1 1 0]

The resulting Euclidean rhythm is E(5,8) = [x . x x . x x .]. This rhythm is illustrated as a polygon (pentagon) in Figure 1 (b). It is another famous rhythm on the world scene. In Cuba it goes by the name of the *cinquillo* and is intimately related to the tresillo [69]. It has been used in jazz throughout the 20*th* century [106], as well as in the *rockabilly* music of the 1950's. For example it is the hand-clapping pattern in Elvis Presley's *Hound Dog* [22]. The cinquillo pattern is also widely used in West African traditional music [105],[125], as well as Egyptian [58] and Korean [65] music.

In the remainder of this section we list some of the most common Euclidean rhythms found in world music. In some cases the Euclidean rhythm is a rotated version of a commonly used rhythm. If a rhythm is a rotated version of another we say that both belong to the same *necklace*. Thus a rhythm necklace is the inter-onset duration interval pattern that disregards the starting point in the cycle. An example of two rhythms that are instances of one and the same necklace is illustrated in Figure 2.

4.1 Periodic Euclidean Rhythms

We will not be concerned with the obvious Euclidean rhythms that occur when k divides without remainder into n. Such perfectly regular rhythms are called *isochronous*. For example, when $k = 3$ and $n = 12$ we obtain E(3,12) = [x . . . x . . . x . . .]. Isochronous rhythms are common all over the planet in traditional, classical, and popular genres of music; they are also periodic: E(3,12) has period 3. In both isochronous and non-isochronous rhythms and meters, slight deviations from perfect regularity are useful as markers

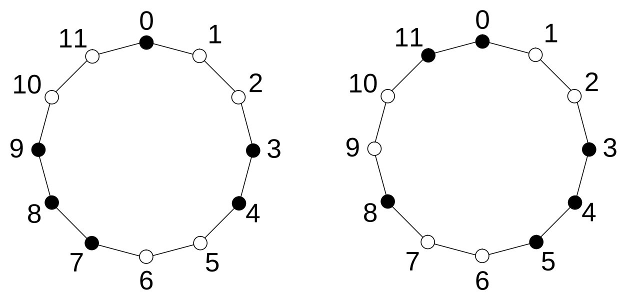

Figure 2: *These two rhythms are instances of one and the same rhythm* necklace.

of higher order periodicities while maintaining an effective distribution of attentional energy [83]. Indeed, there is psychological evidence that such slight deviations from isochrony enhance beat-tracking ability [79]. Therefore we will restrict ourselves to the more interesting non-isochronous Euclidean rhythms.

Furthermore, we will also not be concerned with rhythms that have only one onset. This subfamily of Euclidean rhythms yields the following sequence of rhythms:

E(1,2) = [x .]
E(1,3) = [x . .]
E(1,4) = [x . . .], etc.

Since we restrict ourselves to aperiodic rhythms, we need not enumerate rhythms with different multiples of k and n. For example, multiplying (1,3) by 4 gives (4,12) yielding: E(4,12) = [x . . x . . x . . x . .], which is periodic with four repetitions of E(1,3) = [x . .]. Incidentally, E(4,12) = [x . . x . . x . . x . .] is the (12/8)-time *Fandango* clapping pattern in the Flamenco music of southern Spain, where ‘x’ denotes a loud clap and ‘.’ a soft clap [46], [47].

4.2 Relatively Prime Euclidean Rhythms

The following list of remaining Euclidean rhythms that are found in world music is restricted to values of k and n that are relatively prime. It is perhaps surprizing that such rhythms are not at all rare. Indeed, the following list includes more than 98 such rhythms that I have found so far.

E(2,3) = [x x .] = (12) is a rhythmic pattern of the *Corn Dance* performed at the Santo Domingo Pueblo in New Mexico [110], as well as the *Huapango* rhythm of the Huasteca region of Mexico [122], and the *iambic* rhythm (short-long) from *ars antiqua* [129], traditionally associated with prosody [38]. When started on the second onset as in [x . x], it is a hand-clapping pattern used by the *Bantu* people of Africa [66], as well as the first rhythm taught to beginners of Mandinka drumming [74]. It is also found in Cuba, as for example, the conga rhythm of the (6/8)-time *Swing Tumbao* [73]. It is common in Latin American music, as for example in the *Cueca* [131], and the *coros de clave* [111]. It is common in Arab music, as for example in the *Al Táer* rhythm of Nubia [58]. It is a Tuareg rhythm played on the *tende* drums [135]. In North America it is a drum ostinato found in the *Owl Dance* of the *Flathead* indians of Western Montana in the Unites States [89], and in the Drum Dance of the *Slavey* Indians of Northern Canada [8]. In Greece it is the rhythm of the *Tsamiko* dance [116]. It is also the *trochaic* rhythm (long-short) traditionally associated with prosody [38]. This pattern is also the “ancestral” rhythm obtained from a phylogenetic analysis of Steve Reich’s *Clapping Music* [36]. When started on the silent pulse (anacrusis) as in [. x x], it is used to complement certain African rhythms [31].

$E(2,5)$ = [x . x . .] = (23) is a rhythm found in Greece, Namibia, Rwanda and Central Africa [7]. It is the pattern of the *N-geru* and *Yalli* rhythms used in heroic ballads by the *Tuareg* nomadic people of the Sahara desert [135]. It is the *Rupaka Tisra* tala (of the *Sulaadi* family) of the music of South India [94]. It is also a thirteenth century Persian rhythm called *Khafif-e-ramal* [136], as well as the rhythm of the Macedonian dance *Makedonka* [119]. Tchaikovsky used it as the metric pattern in the second movement of his *Symphony No. 6* [71]. When started on the second onset as in [x . . x .] it is a rhythm found in Central Africa, Bulgaria, Turkey, Turkestan and Norway [7]. It is also the metric pattern of Dave Brubeck's *Take Five* as well as *Mars* from *The Planets* by Gustav Holst [71]. Both starting points determine metric patterns used in Korean music [65], and in classical music composed by Bartok [48]. Both starting points are gong rhythms in their ten-pulse forms (2323) and (3232) in Korean Bhuddist chants [59], and in their ten-pulse form (3223) in Korean Shaman rhythms [64].

$E(2,7)$ = [x . . x . . .] = (34) is the rhythm of the Macedonian *Lesnoto* dance [116]. Started on both onsets, they are two meters used in classical music composed by Bartok [48].

$E(3,4)$ = [x x x .] = (112) is a pattern used in the *Baiaó* rhythm of Brazil [130], a drum rhythm in South Indian classical music [95], as well as the *polos* rhythm of Bali [90]. It is also the *anapest* rhythm (short-short-long) from *ars antiqua* [129], traditionally associated with prosody [38]. When started on the second onset as in [x x . x] it is the *Catarete* rhythm of the indigenous people of Brazil [130], and is also used in ragtime music [68]. It is also the *amphibrach* rhythm (short-long-short) traditionally associated with prosody [38]. When started on the third onset as in [x . x x] it is a *Chingo* rhythm used in Mandinka drumming [74]. It is the archetypal pattern of the *Cumbia* from Colombia [87], as well as a *Calypso* rhythm from Trinidad [53]. It is also a thirteenth century Persian rhythm called *Khalif-e-saghil* [136], as well as the *trochoid choreic* rhythmic pattern of ancient Greece [88]. It is also the *dactil* rhythm (long-short-short) traditionally associated with prosody [38]. When started on the silent note (anacrusis) obtaining [. x x x] it is a popular flamenco rhythm used in the *Taranto*, the *Tiento*, the *Tango*, and the *Tanguillo* [56]. It is also the *Rumba* clapping pattern in flamenco, as well as a second pattern used in the *Baiaó* rhythm of Brazil [130], and a rhythmic pattern played on the *calung panerus* in Indonesian music [57].

$E(3,5)$ = [x . x . x] = (221) is a thirteenth century Indian tala called *Caturthaka* [117]. When started on the second onset, it is a thirteenth century Indian tala called *Dhenki* [118]. it is a thirteenth century Persian rhythm by the name of *Khafif-e-ramal* [136], as well as a Rumanian folk-dance rhythm [102], and the *Sangsa Pyŏlgok* drum pattern in Korean music [65]. It has also been used as a metric pattern in traditional Chinese opera [134].

$E(3,7)$ = [x . x . x . .] = (223) is a rhythm found in Greece, Turkestan, Bulgaria, and Northern Sudan [7]. It is also the Indian *Tritiya* tala [117]. It is the *Dáwer turan* rhythmic pattern of Turkey [58]. It is the *Ruchenitza* rhythm used in a Bulgarian folk-dance [101], as well as the rhythm of the Macedonian dance *Eleno Mome* [119]. It is also the rhythmic pattern of Dave Brubeck's *Unsquare Dance* [24], and Pink Floyd's *Money* [71]. When started on the second onset as in [x . x . . x .] it is a Serbian rhythm [7]. When started on the third onset as in [x . . x . x .] it is a rhythmic pattern found in Greece and Turkey [7]. In Yemen it goes under the name of *Daasa al zreir* [58]. It is also the rhythm of the Macedonian dance *Tropnalo Oro* [119], the rhythm for the Bulgarian *Makedonsko Horo* dance [133], as well as the meter and clapping pattern of the *tīvrā tāl* of North Indian music [32], and the the *Triputa Tisra* tala (of the *Sulaadi* family) of the music of South India [94].

$E(3,8)$ = [x . . x . . x .] = (332) is the ubiquitous Cuban *tresillo* pattern discussed in the preceding [69]; it is a traditional bluegrass banjo rhythm [71], a characteristic bass ostinato in Jamaican *Mento* music [82], a small-drum Burmese rhythmic pattern [13], as well as the *Mai* metal-blade pattern of the *Aka* Pygmies [6]. It is common in West Africa [20] and many other parts of the world such as Greece and Northern Sudan [7]. For example, it is the *Kinka* timeline found in the music of Togo [1]. It is considered to be one of the most important rhythms in Renaissance music by Curt Sachs [113] and Willi Apel [5]. Indeed, it goes all the way back to the Ancient Greeks who called it the *dochmiac* pattern [20]. In India it is one of the 35 sūlādi talas of Karnatak (Carnatic) music [84]. In more recent times it was popular in ragtime music [68] and jazz [39], and most recently it is a popular rhythmic timeline used in electronic dance music [26]. When started on the second onset it is a thirteenth century Indian tala called *Mathya-Tiśra* [118], [94], a timeline used in eastern Angola [78], a drum pattern used in *Samhyŏn Todŭri* Korean instrumental music [65], and a metric pattern used in traditional Chinese opera [134]. Furthermore, it is also found in Bulgaria and Turkey [6]. When started on the third onset it is the *Nandon Bawaa* bell pattern of the *Dagarti* people of northwest Ghana [62], and is also found in Namibia and Bulgaria [6]. All three onsets are used as starting points in accent patterns used in the traditional music of Ghana [99]. When started on the last pulse (anacrusis) as in [. x . . x . . x] it is a bass-drum and cymbal timeline of Burmese music [13].

E(3,10) = [x . . x . . x . . .] = (334), when started on the second onset, is the metric pattern of several Tuareg rhythms played on *tende* drums [135].

E(3,11) = [x . . . x . . . x . .] = (443) is the metric pattern of the *savārī tāl* of North Indian music [32].

E(3,14) = [x . . . . x . . . . x . . .] = (554) is the clapping pattern of the *dhamār tāl* of North Indian music [32].

E(4,5) = [x x x x .] = (1112) is the rhythmic pattern of the *Mirena* rhythm of Greece [58]. When started on the fourth onset, as in [x . x x x] it is the *Tik* rhythm of Greece [58].

E(4,7) = [x . x . x . x] = (2221) is another *Ruchenitza* Bulgarian folk-dance rhythm [101]. When started on the second onset it is the metric pattern played with *ching* (small cymbals) in Thai songs used to accompany dance-dramas dating back to the Ayudhia period (1350-1767) [92]. When started on the third onset it is the *Kalamátianos* Greek dance rhythm [58], as well as the *Shaigie* rhythmic pattern of Nubia [58]. When started on the fourth (last) onset it is the rhythmic pattern of the *Dar daasa al mutawasit* of Yemen [58].

E(4,9) = [x . x . x . x . .] = (2223) is perhaps best known as the *Aksak* rhythm of Turkey [23] (also found in Greece) as well as the rhythm of the Macedonian dance *Kambani Bijat Oro* [119] and the Bulgarian dance *Daichovo Horo* [109]. However, it is also the Indian *gajalila* tala dating back to the thirteenth century [117]. In Bulgarian music fast tunes with this metric pattern are called *Dajchovata* whereas slow tunes with this same pattern are called *Samokovskata* [108]. It is the rhythmic ostinato of a lullaby discovered by Simha Arom in south-western Zaïre [7]. It is the metric pattern used by Dave Brubeck in his well known piece *Rondo a la Turk* [71]. When it is started on the second onset as in [x . x . x . . x .] it is found in Bulgaria and Serbia [7]. When started on the third onset as in [x . x . . x . x .] it is found in Bulgaria and Greece [7]. It is the rhythm of the Macedonian dance *Devojče* [119]. Finally, when started on the fourth onset as in [x . . x . x . x .] it is a rhythm found in Turkey [7], and is the metric pattern of *Strawberry Soup* by Don Ellis [71].

An interesting statistical study by Cler [33] revealed that these four rhythms occur with decreasing frequency in the following order: (2223), (3222), (2232), and (2322).

E(4,11) = [x . . x . . x . . x .] = (3332) is the metric pattern of the *Dhruva Tisra* tala of Southern India [84], It is also used by Frank Zappa in his piece titled *Outside Now* [71]. When it is started on the third onset as in [x . . x . x . . x . .] it is a Serbian rhythmic pattern [7]. We note that according to [94] this is the *Dhruva Tisra* tala of Southern India, contradicting [84]. When it is started on the fourth (last) onset it is the *Daasa* *al kbiri* rhythmic pattern of Yemen [58].

E(4,15) = [x . . . x . . . x . . . x . .] = (4443) is the metric pattern of the *pañcam savārī tāl* of North Indian music [32].

E(5,6) = [x x x x x .] = (11112) yields the *York-Samai* pattern, a popular Arab rhythm [121]. It is also a handclapping rhythm used in the *Al Medēmi* songs of Oman [50].

E(5,7) = [x . x x . x x] = (21211) is the *Nawakhat* pattern, another popular Arab rhythm [121]. In Nubia it is called the *Al Noht* rhythm [58].

E(5,8) = [x . x x . x x .] = (21212) is the ubiquitous Cuban *cinquillo* pattern discussed in the preceding [69], the *Malfuf* rhythmic pattern of Egypt [58], and a small-drum pattern of Burmese music [13]. as well as the Korean *Nong P'yŏn* drum pattern [65]. It is also the *Sangueo* drum rhythm of Venezuela [137]. More recently it is a popular rhythmic timeline used in electronic dance music [26]. When started on the second pulse (anacrusis) as in [. x x . x x . x] it is a bass-drum pattern used in music from Burma [13]. When started on the second onset it is a popular Latin-American rhythm used in many styles of music such as the *Tango* of Argentina [53], the *Merengue* of the Dominican Republic [68], and the *Calypso* of Trinidad [68]. It is also a popular Middle East rhythm [133] used in the *Maksum* of Egypt [58], and is a thirteenth century Persian rhythm called the *Al-saghil-al-sani* [136]. In addition it is a rhythm used in Sub-Saharan Africa, for example the *Timini* of Senegal, and the *Adzogbo* dance rhythm of Benin [30]. When it is started on the third onset it is the *Müsemmen* rhythm of Turkey [16]. When it is started on the fourth onset it is the *Kromanti* rhythm of Surinam and the *Gabada* timeline found in the music of West Africa [1]. When started on the fifth onset it is a rhythm used in the Cuban habanera as well as ragtime music [68].

E(5,9) = [x . x . x . x . x] = (22221) is a popular Arab rhythm called *Agsag-Samai* [121], as well as the *Kaarika* tala of the *Carnatic* classical music of India [94]. When started on the second onset, it is a drum pattern used by the *Venda* in South Africa [105],[19], as well as a Rumanian folk-dance rhythm [102]. It is also the rhythmic pattern of the *Sigaktistos* rhythm of Greece [58], and the *Samai aktsak* rhythm of Turkey [58]. When started on the third onset it is the rhythmic pattern of the *Nawahiid* rhythm of Turkey [58].

E(5,11) = [x . x . x . x . x . .] = (22223) is the metric pattern of the Savārī tala used in the Hindustani music of India [84]. It is also a rhythmic pattern used in Bulgaria and Serbia [7]. In Bulgaria is is used in the *Kopanitsa* [109]. This metric pattern has been used by Moussorgsky in *Pictures at an Exhibition* [71]. When started on the third onset it is the rhythm of the Macedonian dance *Kalajdzijsko Oro* [119], and it appears in Bulgarian music as well [7].

E(5,12) = [x . . x . x . . x . x .] = (32322) is a common rhythm played in the Central African Republic by the *Aka* Pygmies [6], [27], [29]. It is also the *Venda* clapping pattern of a South African children's song [101], and a rhythm pattern used in Macedonia [7]. When started on the second onset it is the *Columbia* bell pattern popular in Cuba and West Africa [73]; it is a drumming pattern used in the *Chakacha* dance of Kenya [12], and a metric pattern used in Macedonia [7]. When started on the third onset it is the *Bemba* bell pattern used in Northern Zimbabwe [101], and the rhythm of the Macedonian dance *Ibraim Odža Oro* [119]. When started on the fourth onset it is a clapping pattern used widely in West, Central, and East Africa [66], [2]. It is also the *Fume Fume* bell pattern popular in West Africa [73], a bell pattern used in Zaire [78], and a metric pattern used in the former Yugoslavia [7]. Finally, when started on the fifth onset it is the *Salve* bell pattern used in the Dominican Republic in a rhythm called *Canto de Vela* in honor of the Virgin Mary [54], a rhythmic pattern used by the *Tuareg* nomadic people of the Sahara [135], as well as the drum rhythmic pattern of the Moroccan *Al Kudám* [58].

$E(5,13)$ = [x . . x . x . . x . x . .] = (32323) is a Macedonian rhythm which is also played by starting it on the fourth onset as follows: [x . x . . x . . x . x . .] [7].

$E(5,16)$ = [x . . x . . x . . x . . x . . .] = (33334) is a popular rhythmic timeline used in electronic dance music [26]. It is also the *Bossa-Nova* rhythm necklace of Brazil. The actual Bossa-Nova rhythm usually starts on the third onset as follows: [x . . x . . x . . . x . . x . .] [125]. However, there are other starting places as well, as for example [x . . x . . x . . x . . . x . .] [14].

$E(6,7)$ = [x x x x x x .] = (111112) is the rhythmic pattern of the *Póntakos* rhythm of Greece when started on the sixth (last) onset [58].

$E(6,13)$ = [x . x . x . x . x . x . .] = (222223) is the rhythm of the Macedonian dance *Mama Cone pita* [119]. When started on the third onset it is the rhythm of the Macedonian dance *Postupano Oro* [119], as well as the *Krivo Plovdivsko Horo* of Bulgaria [109].

$E(7,8)$ = [x x x x x x x .] = (1111112), when started on the seventh (last) onset, is a typical rhythm played on the *Bendir* (frame drum), and used in the accompaniment of songs of the *Tuareg* people of Libya [121]. When started on the eighth (last) pulse (anacrusis), it is a rhythmic pattern played on the *calung panerus* in Indonesian music [57].

$E(7,9)$ = [x . x x x . x x x] = (2112111) is the *Bazaragana* rhythmic pattern of Greece [58].

$E(7,10)$ = [x . x x . x x . x x] = (2121211) is the *Lenk fahhte* rhythmic pattern of Turkey [58].

$E(7,12)$ = [x . x x . x . x x . x .] = (2122122) is used by the *Ashanti* people of Ghana in several rhythms [101]. It is used in the *Dunumba* rhythm of Guinea [60], and by the *Akan* people of Ghana [98] as a juvenile song rhythm. It is also a pattern used by the *Bemba* people of Northern Zimbabwe, where it is either a hand-clapping pattern, or played by chinking pairs of axe-blades together [66], [67].

When started on the second onset it is a hand-clapping pattern used in Ghana [101], Southern Africa [66], and Tanzania [132]. It is also played as a secondary bell pattern in the Cuban *Bembé* rhythm on a low pitched bell [93].

When started on the third onset it is the most important rhythm in Sub-Saharan Africa. It is worth noting that it is the same pattern as the pitch pattern of the major diatonic scale. This rhythm, denoted by [x . x . x x . x . x . x], is probably the most (internationally) well known of all the African timelines. Indeed, the master drummer Desmond K. Tai has called it the Standard Pattern [67], and it also goes by the name African Signature Tune [1]. In West Africa it is found under various names among the Ewe and Yoruba peoples [101]. In Ghana it is the timeline played in the *Agbekor* dance rhythm found along the southern coast of Ghana [30], in the *Agbadza* [1], as well as in the *Bintin* rhythm [93] (see also chapter 22 of Collins [37]). Among the *Ewe* this rhythm is also a bell pattern used in the *Adzogbo* dance music [81]. Furthermore, among the Ewe people of Ghana there is a unique rhythm, for five bells only, called the *Gamamla* [73]. The standard pattern is one of the five Gamamla bell patterns played on the *Gankogui* (a double bell), with the first note played on the low pitched bell, and the other six on the high pitched bell. The same is done in the *Sogba* and *Sogo* rhythms of the Ewe people. It is played in the *Zebola* rhythm of the *Mongo* people of Congo, and in the *Tiriba* and *Liberté* rhythms of Guinea [60]. This bell pattern is equally widespread in America. In Cuba it is the principal bell pattern played on the *guataca* or hoe blade in the *Batá* rhythms [91]. For example, it is used in the *Columbia de La Habana*, the *Bembé*, the *Chango*, the *Eleggua*, the *Imbaloke*, and the *Palo*. The word *palo* in Spanish means stick and refers also to the sugar cane. The Palo rhythm was played during the cutting of sugar cane in Cuba. The pattern is also used in the *Guiro*, a Cuban folkloric rhythm [73]. In Haiti it is called the *Ibo* [93]. In Brazil it goes by the name of *Behavento* [93]. In North

America this rhythm is sometimes called the *short* African bell pattern [49].

When started on the fourth onset it is a rhythm found in Northern Zimbabwe called the *Bemba* [101] (not to be confused with the *Bembé* from Cuba), and played using axe blades [66]. In Cuba it is the bell pattern of the *Sarabanda* rhythm associated with the Palo Monte cult [132].

When started on the fifth onset it is the *Bondo* bell pattern played with metal strips by the *Aka* pygmies of Central Africa [6].

When started on the sixth onset it is a bell pattern found in several places in the Caribbean, including Curaçao, where it is used in a rhythm by the name *Tambú* [112]. Originally this rhythm was played with only two instruments: a drum and a metallophone called the *heru*. Note that the word Tambú sounds like *tambor*, the Spanish word for drum, and heru sounds like *hierro*, the Spanish word for iron. This bell pattern is also common in West Africa and Haiti [101]. In Central Africa it is called the *Muselemeka* timeline [78], and in North America it is sometimes called the *long* African bell pattern [49]. Strangely enough Changuito uses this pattern in what he calls the *Bembé*, thus at odds with what everyone else calls *Bembé*, namely the pattern [x . x . x x . x . x . x] [104].

When started on the seventh onset it is a *Yoruba* bell pattern of Nigeria, a *Babenzele* pattern of Central Africa, and a *Mende* pattern of Sierra Leone [123]. Among the Yoruba people it is also called the *konkonkolo* pattern [1].

When started on the tenth pulse (anacrusis) it is a palitos pattern used in the *Columbia* style of Cuban *Rumba* dance music [42].

E(7,15) = [x . x . x . x . x . x . x . .] = (2222223) is a Bulgarian rhythm when started on the third onset [7].

E(7,16) = [x . . x . x . x . . x . x . x .] = (3223222) is a *Samba* rhythm necklace from Brazil. The actual Samba rhythm is [x . x . . x . x . x . . x . x .] obtained by starting E(7,16) on the last onset, and it coincides with a Macedonian rhythm [7]. When E(7,16) is started on the fifth onset it is a clapping pattern from Ghana [101]. When it is started on the second onset it is a rhythmic pattern found in the former Yugoslavia [7]. Furthermore, when it is started at the midpoint between the third and fourth onsets (anacrusis) it is the *Partido Alto* rhythm used in the *Pagode* style of *Samba* music in Brazil [97].

E(7,17) = [x . . x . x . . x . x . . x . x .] = (3232322) is a Macedonian rhythm when started on the second onset [119].

E(7,18) = [x . . x . x . . x . x . . x . x . .] = (3232323) is a Bulgarian rhythmic pattern [7].

E(8,17) = [x . x . x . x . x . x . x . x . .] = (22222223) is a Bulgarian rhythmic pattern which is also started on the fifth onset [7].

E(8,19) = [x . . x . x . x . . x . x . x . . x .] = (32232232) is a Bulgarian rhythmic pattern when started on the second onset [7].

E(9,13) = [x . x x . x x . x x . x x] = (212121211), when started on the last onset, is the Bohlen-Pierce scale [76].

E(9,14) = [x . x x . x x . x x . x x .] = (212121212), when started on the second onset, is the rhythmic pattern of the *Tsofyan* rhythm of Algeria [58].

E(9,16) = [x . x x . x . x . x x . x . x .] = (212221222) is a rhythm necklace used in the Central African Republic [6]. When started on the first onset it is a *palitos* rhythm used in the Cuban *Rumba* [42], [15]. When it is started on the second onset it is a bell pattern of the *Luba* people of Congo [103]. When it is started on the fourth onset it is a rhythm played in West and Central Africa [69], as well as a cow-bell pattern in the Brazilian *Samba* [120]. When the referential beat is the seventh onset it is the *Kachacha* timeline used in Central Africa [78]. When it is started on the penultimate onset it is the bell pattern of the *Ngbaka-Maibo* rhythms of the Central African Republic [6].

E(9,20) = [x . . x . x . x . x . . x . x . x . x .] = (322232222), when started on the last onset, is Balzano's 20-fold scale [76].

E(9,22) = [x . . x . x . . x . x . . x . x . . x . x .] = (323232322) is a Bulgarian rhythmic pattern when started on the second onset [7].

E(9,23) = [x . . x . x . . x . x . . x . x . . x . x . .] = (323232323) is a Bulgarian rhythm [7].

E(11,12) = [x x x x x x x x x x x .] = (11111111112), when started on the second onset, is the drum pattern of the *Rahmāni* (a cylindrical double-headed drum) used in the *Sōt silām* dance from *Mirbāt* in the South of Oman [50].

E(11,20) = [x . x x . x . x . x . x x . x . x . x .] = (21222212222), when started on the ninth onset, is the eleven-note scale considered to be the correct analogue of the seven-note diatonic scale in the twelve-tone system [139].

E(11,24) = [x . . x . x . x . x . x . . x . x . x . x . x .] = (32222322222) is a rhythm necklace of the *Aka* Pygmies of Central Africa [6]. It is usually started on the seventh onset. When started on the second onset it is a Bulgarian rhythm [7].

E(13,24) = [x . x x . x . x . x . x . x x . x . x . x . x .] = (2122222122222) is another rhythm necklace of the *Aka* Pygmies of the upper *Sangha* [6]. When started on the penultimate (12-th) onset it is the *Bobangi* metal-blade pattern used by the *Aka* Pygmies.

E(15,34) = [x . . x . x . x . x . . x . x . x . x . . x . x . x . x . . x . x .] = (322232223222322) is a Bulgarian rhythmic pattern when started on the second to last onset [7].

E(19,30) = [x . x x . x . x x . x x . x x . x . x x . x x . x x . x . x] = (2122121212212121221), when started on the seventeenth onset, is the nineteen-note scale generated by a group-theoretic method [139].

5. Aksak Rhythms

Euclidean rhythms are closely related to a family of rhythms known as *aksak* rhythms, which have been studied from the combinatorial point of view for some time now [23], [33], [7]. B´ela Bart´ok [11] and Constantin Brăiloiu [23], respectively, have used the terms *Bulgarian rhythm* and *aksak* to refer to those meters which use units of durations 2 and 3, and no other durations. Furthermore, the rhythm or meter must contain at least one duration of length 2 and at least one duration of length 3. Arom [7] referes to these durations as *binary cells* and *ternary cells*, respectively.

Arom [7] has generated an inventory of all the theoretically possible *aksak* rhythms for values of n ranging from 5 to 29, as well as a list of those that are actually used in traditional world music. He has also proposed a classifi cation of these rhythms into several classes, based on structural and numeric properties. Three of his classes are considered here: *authentic-aksaks*, *quasi-aksaks*, and *pseudo-aksaks*.

- An *aksak* rhythm is *authentic* if n is a *prime* number.
- An *aksak* rhythm is *quasi-aksak* if n is an *odd* number that is not prime.
- An *aksak* rhythm is *pseudo-aksak* if n is an *even* number.

A quick perusal of the Euclidean rhythms listed in the preceding reveals that *aksak* rhythms are well represented. Indeed, all three of Arom's classes (authentic, quasi-aksak, and pseudo-aksak) make their appearance. There is a simple characterization of those Euclidean rhythms that are *aksak*. From the iterative subtraction algorithm of Bjorklund it follows that if $n = 2k$ all cells are binary (duration 2). Similarly, if $n = 3k$ all cells are ternary (duration 3). Therefore, if we want to ensure that the Euclidean rhythm contains both binary and ternary cells, and no other durations, it follows that n must fall in between. More precisely:

Observation 1 A Euclidean rhythm is *aksak* provided that $2k < n < 3k$.

Of course, not all *aksak* rhythms are Euclidean. Consider the Bulgarian rhythm with interval sequence (3322) [7], which is also the metric pattern of *Indian Lady* by Don Ellis [71]. Here $k = 4$ and $n = 10$, and E(4,10) = [x . . x . x . . x .] or (3232), a periodic rhythm.

The following Euclidean rhythms are *authentic aksak*:

E(2,5) = [x . x . .] = (23) (classical music, jazz, Greece, Macedonia, Namibia, Persia, Rwanda), (*authentic aksak*).
E(3,7) = [x . x . x . .] = (223) (Bulgaria, Greece, Sudan, Turkestan), (*authentic aksak*).
E(4,11) = [x . . x . . x . . x .] = (3332) (Southern India rhythm), (Serbian necklace), (*authentic aksak*).
E(5,11) = [x . x . x . x . x . .] = (22223) (classical music, Bulgaria, Northern India, Serbia), (*authentic aksak*).
E(5,13) = [x . . x . x . . x . x . .] = (32323) (Macedonia), (*authentic aksak*).
E(6,13) = [x . x . x . x . x . x . .] = (222223) (Macedonia), (*authentic aksak*).
E(7,17) = [x . . x . x . . x . x . . x . x .] = (3232322) (Macedonian necklace), (*authentic aksak*).
E(8,17) = [x . x . x . x . x . x . x . x . .] = (22222223) (Bulgaria), (*authentic aksak*).
E(8,19) = [x . . x . x . x . . x . x . x . . x .] = (32232232) (Bulgaria), (*authentic aksak*).
E(9,23) = [x . . x . x . . x . x . . x . x . . x . x . .] = (323232323) (Bulgaria), (*authentic aksak*).

The following Euclidean rhythms are *quasi-aksak*:

E(4,9) = [x . x . x . x . .] = (2223) (Greece, Macedonia, Turkey, Zaïre), (*quasi-aksak*).
E(7,15) = [x . x . x . x . x . x . x . .] = (2222223) (Bulgarian necklace), (*quasi-aksak*).

The following Euclidean rhythms are *pseudo-aksak*:

E(3,8) = [x . . x . . x .] = (332) (Central Africa, Greece, India, Latin America, West Africa, Sudan), (*pseudo-aksak*).
E(5,12) = [x . . x . x . . x . x .] = (32322) (Macedonia, South Africa), (*pseudo-aksak*).
E(7,16) = [x . . x . x . x . . x . x . x .] = (3223222) (Brazilian, Macedonian, West African necklaces), (*pseudo-aksak*).
E(7,18) = [x . . x . x . . x . x . . x . x . .] = (3232323) (Bulgaria), (*pseudo-aksak*).
E(9,22) = [x . . x . x . . x . x . . x . x . . x . x .] = (323232322) (Bulgarian necklace), (*pseudo-aksak*).
E(11,24) = [x . . x . x . x . x . x . . x . x . x . x . x .] = (32222322222) (Central African and Bulgarian necklaces), (*pseudo-aksak*).
E(15,34) = [x . . x . x . x . x . . x . x . x . x . . x . x . x . x . . x . x .] = (322232223222322) (Bulgarian necklace), (*pseudo-aksak*).

6. Calculating Leap Years in Calendar Design

For thousands of years human beings have observed and measured the time it takes between two consecutive sunrises, between two consecutive full moons, and between two consecutive spring seasons. These measurements inspired different cultures to design calendars in different ways [9], [107]. Let T_y denote the duration of one revolution of the earth around the sun, more commonly known as a year. Let T_d denote the duration of one complete rotation of the earth, more commonly known as a day. The values of T_y and T_d are of course continually changing, since the universe is continually reconfiguring itself. However the ratio T_y/T_d is approximately 365.242199..... It is very convenient therefore to make a year last 365 days. The problem that arizes both for history and for predictions of the future, is that after a while the 0.242199..... starts to contribute to a large error. One simple solution is to add one extra day every 4 years: the so-called Julian calendar. A day with one extra day is called a leap year. But this assumes that a year is 365.25 days long, which is still slightly greater than 365.242199...... So now we have an error in the opposite direction albeit smaller. One solution to this problem is the Gregorian calendar [115]. The Gregorian calendar defines a leap year as one divisible by 4, except not those divisible by 100, except not those divisible by 400. With this rule a year becomes 365 + 1/4 - 1/100 + 1/400 = 365.2425 days long, not a bad approximation. Another solution is provided by the Jewish calendar which uses the idea of cycles [9]. Here a regular year has 12 months and a leap year has 13 months. The cycle has 19 years including 7 leap years. The 7 leap years must be distributed as evenly as possible in the cycle of 19. The cycle is assumed to start with Creation as year 1. Then, when the year number is divided by 19, the remainder indicates the resulting position in the cycle. The leap years are 3, 6, 8, 11, 14, 17, and 19. For example, the year $5765 = 303 \times 19 + 8$ and so is a leap year. The year 5766, which begins at sundown on the Gregorian date of October 3, 2005, is $5766 = 303 \times 19 + 9$, and is therefore not a leap year. Applying Bjorklund's algorithm to the integers 7 and 19 yields E(7,19) = [x . . x . x . . x . . x . x . . x . .]. If we start this rhythm at the 7th pulse we obtain the pattern [. . x . . x . x . . x . . x . . x . x], which describes precisely the leap year pattern 3, 6, 8, 11, 14, 17, and 19 of the Jewish calendar. Therefore the leap-year pattern of the Jewish calendar is a Euclidean necklace.

The Islamic calendar is based on the time between two successive new moons (*lunations*), and one year is defined as twelve lunations [107]. This method gives approximately 10632 days every 30 years, in which 11 leap years are needed. The common approximations used in the Islamic calendar put leap years at positions 2, 5, 7, 10, 13, 16, 18, 21, 24, 26, and 29 in the cycle. Applying Bjorklund's algorithm to the integers 11 and 30 yields E(11,30) = [x . . x . x . . x . . x . . x . x . . x . . x . . x . x . .]. If we start this rhythm on the 11th pulse we obtain the pattern [. x . . x . x . . x . . x . . x . x . . x . . x . x . . x .] which describes precisely the leap year pattern 2, 5, 7, 10, 13, 16, 18, 21, 24, 26, and 29 of the Islamic calendar. Therefore the leap-year pattern of the Islamic calendar also is a Euclidean necklace.

7. Drawing Digital Straight Lines

Euclidean rhythms and necklace patterns also appear in the computer graphics literature on drawing digital straight lines [72]. The problem here consists of efficiently converting a mathematical straight line segment defined by the x and y integer coordinates of its endpoints, to an ordered sequence of pixels that most faithfully represents the given straight line segment. Figure 3 illustrates an example of a digital straight line (shaded pixels) determined by the two given endpoints p and q. All the pixels intersected by by the segment (p, q) are shaded. If we follow either the lower or upper boundary of the shaded pixels from left to right we obtain the interval sequences (43333) or (33334), respectively. Note that the upper pattern corresponds to E(5,16), a *Bossa-Nova* variant. Indeed, Harris and Reingold [61] show that the well-known Bresenham algorithm [21] for drawing digital straight lines on a computer screen is implemented by the Euclidean Algorithm.

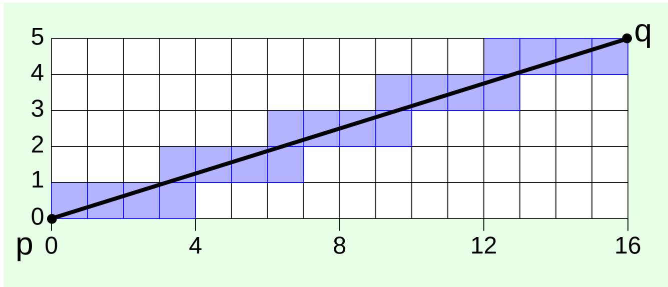

Figure 3: *The shaded pixels form a digital straight line determined by the points p and q.*

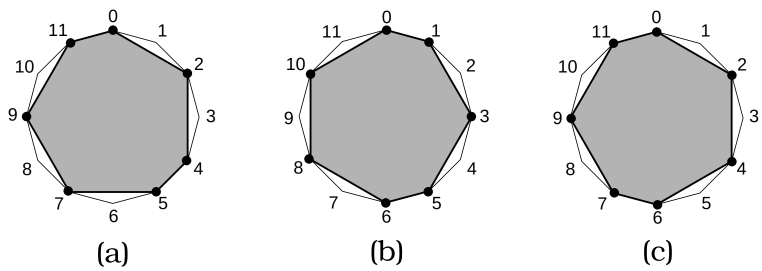

Figure 4: *Two right-rotations of the* Bemb´e *string: (a) the* Bemb´e, *(b) rotation by one unit, (c) rotation by seven units.*

8. Sturmian Words, Markoff Numbers, and Two-distance Sequences

The Euclidean rhythms considered here are known by many different names in several areas of mathematics. In the algebraic combinatorics of words they are called *Sturmian* words [85]. They are called *two-distance* sequences by Lunnon and Pleasants [86], and *Beatty* sequences by de Bruijn [43], [44]. See also the geometry of Markoff numbers [114].

9. Euclidean Strings

In the study of the combinatorics of words and sequences, there exists a family of strings called Euclidean strings [51]. In this section we explore the relationship between Euclidean strings and Euclidean rhythms. We use the same terminology and notation introduced in [51].

Let $P = (p_0, p_1, ..., p_{n-1})$ denote a string of non-negative integers. Let $\rho(P)$ denote the right rotation of P by one position, i.e., $\rho(P) = (p_{n-1}, p_0, p_1, ..., p_{n-2})$, and let $\rho^d(P)$ denote the right rotation of P by d positions. If P is considered as a cyclic string, a right rotation corresponds to a clockwise rotation. Figure 4 illustrates the $\rho(P)$ operator with P equal to the *Bembé* bell-pattern of West Africa [126]. Figure 4 (a) shows the *Bembé* bell-pattern, Figure 4 (b) shows $\rho(P)$, which is a hand-clapping pattern from West Africa [101], and Figure 4 (c) shows $\rho^7(P)$, which is the *Tambú* rhythm of Curaçao [112].

Ellis et al., [51] defi ne a string $P = (p_0, p_1, ..., p_{n-1})$ as a *Euclidean string* if increasing p_0 by one, and decreasing p_{n-1} by one, yields a new string, denoted by $\tau(P)$, that is a rotation of P, i.e., P and $\tau(P)$

are instances of one and the same necklace. Therefore, if we represent rhythms as binary sequences, Euclidean rhythms cannot be Euclidean strings because by virtue of the Euclidean algorithm employed, all Euclidean rhythms begin with a 'one'. Increasing p_0 by one makes it a 'two', which is not a binary string. Therefore, to explore the relationship between Euclidean strings and Euclidean rhythms, we will represent rhythms by their adjacent-inter-onset-duration-interval-vectors, (interval-vectors for short) which are also strings of non-negative integers. As an example, consider the *Aksak* rhythm of Turkey [23] given by E(4,9) = [x . x . x . x . .]. In interval-vector notation we have that E(4,9) = (2223). Now $\tau(2223) = (3222)$, which is a rotation of E(4,9), and thus (2223) is a Euclidean string. Indeed, for P=E(4,9), $\tau(P) = \rho^3(P)$. As a second example. consider the West African clapping-pattern shown in Figure 4 (b) given by P=(1221222). We have that $\tau(P) =$ (2221221) $= \rho^6(P)$, the pattern shown in Figure 4 (c), which also happens to be the mirror image of P about the $(0, 6)$ axis. Therefore P is a Euclidean string. However, note that P is not a Euclidean rhythm. Nevertheless, P is a rotation of the Euclidean rhythm E(7,12)=(2122122).

Ellis et al., [51] have many beautiful results about Euclidean strings. They show that Euclidean strings exist if, and only if, n and $(p_0 + p_1 + ... + p_{n-1})$ are relatively prime numbers, and that when they exist they are unique. They also show how to construct Euclidean strings using an algorithm that has the same structure as the Euclidean algorithm. In addition they relate Euclidean strings to many other families of sequences studied in the combinatorics of words [3], [85].

Note that in the operational definition of Euclidean strings increasing p_0 by one, and decreasing p_{n-1} by one in the interval vector representation of a rhythm is tantamount to performing a swap operation between the last and first pulses (exchanging a one and a zero) of the rhythm expressed in binary (or box) notation. In this setting, obtaining a rotation of a rhythm in the definition of Euclidean strings is a special case of the notion of *P-cycles* studied in music theory [76]. In a P-cycle the swap operation, when applied repeatedly on a suitable onset of a rhythm, will generate *all* rotations of the rhythm.

Let $R(P)$ denote the reversal (or mirror image) of P, i.e., $R(P) = (p_{n-1}, p_{n-2}, ..., p_1, p_0)$. For example, for the *Aksak* rhythm where P = (2223), we obtain that $R(P)$ = (3222), i.e., $R(P)$ implies playing the rhythm P backwards by starting at the same onset. Now we may determine which of the Euclidean rhythms used in world music listed in the preceding, are Euclidean strings or *reverse* Euclidean strings. The length of a Euclidean string is defined as the number of integers it has. This translates in the rhythm domain to the number of onsets a rhythm contains. Furthermore, strings of length one are Euclidean strings, trivially. Therefore all the trivial Euclidean rhythms with only one onset, such as E(1,2) = [x .] = (2), E(1,3) = [x . .] = (3), and E(1,4) = [x . . .] = (4), etc., are both Euclidean strings as well as reverse Euclidean strings. In the lists that follow the Euclidean rhythms are shown in their box-notation format as well as in the interval-vector representation. The styles of music that use these rhythms is also included. Finally, if only a rotated version of the Euclidean rhythm is played, then it is still included in the list but referred to as a necklace.

The following Euclidean rhythms are Euclidean strings:

E(2,3) = [x x .] = (12) (West Africa, Latin America, Nubia, Northern Canada).
E(2,5) = [x . x . .] = (23) (classical music, jazz, Greece, Macedonia, Namibia, Persia, Rwanda), (*authentic aksak*).
E(2,7) = [x . . x . . .] = (34) (classical music).
E(3,4) = [x x x .] = (112) (Brazil, Bali rhythms), (Colombia, Greece, Spain, Persia, Trinidad necklaces).
E(3,7) = [x . x . x . .] = (223) (Bulgaria, Greece, Sudan, Turkestan), (*authentic aksak*).
E(3,10) = [x . . x . . x . . .] = (334) (Tuareg necklace).
E(4,5) = [x x x x .] = (1112) (Greece).
E(4,9) = [x . x . x . x . .] = (2223) (Greece, Macedonia, Turkey, Zaïre), (*quasi-aksak*).
E(5,6) = [x x x x x .] = (11112) (Arab).

E(5,11) = [x . x . x . x . x . .] = (22223) (classical music, Bulgaria, Northern India, Serbia), (*authentic aksak*).
E(5,16) = [x . . x . . x . . x . . x . . . .] = (33334) (Brazilian, West African necklaces).
E(6,7) = [x x x x x x .] = (111112) (Greek necklace)
E(6,13) = [x . x . x . x . x . x . .] = (222223) (Macedonia), (*authentic aksak*).
E(7,8) = [x x x x x x x .] = (1111112) (Libyan necklace).
E(7,15) = [x . x . x . x . x . x . x . .] = (2222223) (Bulgarian necklace), (*quasi-aksak*).
E(8,17) = [x . x . x . x . x . x . x . x . .] = (22222223) (Bulgaria), (*authentic aksak*).

The following Euclidean rhythms are reverse Euclidean strings:

E(3,5) = [x . x . x] = (221) (India), (Korean, Rumanian, Persian necklaces).
E(3,8) = [x . . x . . x .] = (332) (Central Africa, Greece, India, Latin America, West Africa, Sudan), (*pseudo-aksak*).
E(3,11) = [x . . . x . . . x . .] = (443) (North India).
E(3,14) = [x . . . . x . . . . x . . .] = (554) (North India).
E(4,7) = [x . x . x . x] = (2221) (Bulgaria).
E(4,11) = [x . . x . . x . . x .] = (3332) (Southern India rhythm), (Serbian necklace), (*authentic aksak*).
E(4,15) = [x . . . x . . . x . . . x . .] = (4443) (North India).
E(5,7) = [x . x x . x x] = (21211) (Arab).
E(5,9) = [x . x . x . x . x] = (22221) (Arab).
E(5,12) = [x . . x . x . . x . x .] = (32322) (Macedonia, South Africa), (*pseudo-aksak*).
E(7,9) = [x . x x x . x x x] = (2112111) (Greece).
E(7,10) = [x . x x . x x . x x] = (2121211) (Turkey).
E(7,16) = [x . . x . x . x . . x . x . x .] = (3223222) (Brazilian, Macedonian, West African necklaces), (*pseudo-aksak*).
E(7,17) = [x . . x . x . . x . x . . x . x .] = (3232322) (Macedonian necklace), (*authentic aksak*).
E(9,22) = [x . . x . x . . x . x . . x . x . . x . x .] = (323232322) (Bulgarian necklace), (*pseudo-aksak*).
E(11,12) = [x . x x x x x x x x x x] = (11111111112) (Oman necklace).
E(11,24) = [x . . x . x . x . x . x . . x . x . x . x . x .] = (32222322222) (Central African and Bulgarian necklaces), (*pseudo-aksak*).

The following Euclidean rhythms are neither Euclidean nor reverse Euclidean strings:

E(5,8) = [x . x x . x x .] = (21212) (Egypt, Korea, Latin America, West Africa).
E(5,13) = [x . . x . x . . x . x . .] = (32323) (Macedonia), (*authentic aksak*).
E(7,12) = [x . x x . x . x x . x .] = (2122122) (West Africa), (Central African, Nigerian, Sierra Leone necklaces).
E(7,18) = [x . . x . x . . x . x . . x . x . .] = (3232323) (Bulgaria), (*pseudo-aksak*).
E(8,19) = [x . . x . x . x . . x . x . x . . x .] = (32232232) (Bulgaria), (*authentic aksak*).
E(9,14) = [x . x x . x x . x x . x x .] = (212121212) (Algerian necklace).
E(9,16) = [x . x x . x . x . x x . x . x .] = (212221222) (West and Central African, and Brazilian necklaces).
E(9,23) = [x . . x . x . . x . x . . x . x . . x . x . .] = (323232323) (Bulgaria), (*authentic aksak*).
E(13,24) = [x . x x . x . x . x . x . x x . x . x . x . x .] = (2122222122222) (Central African necklace).
E(15,34) = [x . . x . x . x . x . . x . x . x . x . . x . x . x . x . . x . x .] = (322232223222322) (Bulgarian necklace), (*pseudo-aksak*).

10. Concluding Remarks

A family of musical rhythms is described, dubbed Euclidean rhythms, which are obtained by using Bjorklund's sequence generation algorithm, which has the same structure as the Euclidean algorithm.

As mentioned in the introduction, during the past thirty years a number of researchers have approached the study of rhythmic timelines using *generative* methods, notably Kubik [77], Locke [80], Pressing [101], Rahn [105], [106], Anku [4], Toussaint [125], [126], [127], [128], and Agawu [1]. Agawu [1] provides an in-depth analysis of these methods applied to African timelines. On the other hand, the Euclidean algorithm exposed here is a mathematical model of rhythmic timeline generation that applies to music from all over the world. A notable exception are the Indian *talas*, which tend to have longer timelines than other music (as many as 128 beats per cycle), and therefore use a greater variety of duration intervals, thus violating the maximal evenness of Euclidean rhythms. A rhythm is said to be maximally even if its representation on the circle of time maximizes the sum of its pairwise inter-onset straight line distances [34], [35]. For example, of the 35 *Sulaadi* talas only 4 are maximally even, and of the 108 *Astottara Sata* talas only 9 are maximally even [94]. It has been shown by Demaine et al., [45] that a rhythm is maximally even if and only if it is Euclidean, or a rotation of a Euclidean rhythm. Euclidean (maximally even) rhythms have many interesting mathematical and musical properties. For example, the complement of a Euclidean rhythm is also Euclidean [124].

The three groups of Euclidean rhythms listed in the preceding section reveal a tantalizing pattern. The Euclidean rhythms that are favoured in classical music and jazz are also Euclidean strings (the first group). Furthermore, this group is not popular in African music. The Euclidean rhythms that are neither Euclidean strings nor reverse Euclidean strings (group three) fall into two categories: those consisting of interval lengths '1' and '2', and those consisting of interval lengths '2' and '3'. The latter group is used only in Bulgaria, and the former is used in Africa. Finally, the Euclidean rhythms that are reverse Euclidean strings (the second group) appear to have a much wider appeal. Finding musicological explanations for the preferences apparent in these mathematical properties raizes interesting ethnomusicological questions.

Acknowledgements

I would like to thank Simha Arom for providing me with a copy of his paper on the classification of *aksak* rhythms [7], which provided several examples of Euclidean rhythms of which I was not aware. Thanks to Jay Rahn for copies of his enlightening papers on African timelines. Thanks to Jeff Erickson for bringing to my attention the wonderful paper by Mitchell Harris and Ed Reingold on digital line drawing, leap year calculations and Euclid's algorithm. I am also grateful to Marcia Ascher for pointing out that the pattern of leap years in the Jewish calendar is a Euclidean necklace.

References

[1] Kofi Agawu. Structural analysis or cultural analysis? Competing perspectives on the "standard pattern" of West African rhythm. *Journal of the American Musicological Society*, 59(1):1–46, 2006.

[2] Samuel Akpabot. Theories on African music. *African Arts*, 6(1):59–62, Autumn 1972.

[3] J.-P. Allouche and J. O. Shallit. *Automatic Sequences*. Cambridge University Press, Cambridge, England, 2002.

[4] Willie Anku. Circles and time: A theory of structural organization of rhythm in African music. *Music Theory Online*, 6(1), January 2000.

[5] Willi Apel. Vier plus vier = drei plus drei plus zwei. *Acta Musicologica*, 32(Fasc. 1):29–33, January-March 1960.

[6] Simha Arom. *African Polyphony and Polyrhythm*. Cambridge University Press, Cambridge, England, 1991.

[7] Simha Arom. L'aksak: Principes et typologie. *Cahiers de Musiques Traditionnelles*, 17:12–48, 2004.

[8] Michael I. Asch. Social context and the musical analysis of Slavey drum dance songs. *Ethnomusicology*, 19(2):245–257, May 1975.

[9] Marcia Ascher. *Mathematics Elsewhere: An Exploration of Ideas Across Cultures*. Princeton University Press, Princeton and Oxford, 2002.

[10] Anthony Ashton. *Harmonograph–A Visual Guide to the Mathematics of Music*. Walker and Company, New York, 2003.

[11] B´ela Bart´ok. Ce qu'on appelle le rythme bulgare. In *Musique de la vie*, pages 142–155, Stock, Paris, 1981.

[12] Gregory Barz. *Music in East Africa*. Oxford University Press, Oxford, England, 2004.

[13] Judith Becker. Percussive patterns in the music of mainland southeast Asia. *Ethnomusicology*, 12(2):173–191, May 1968.

[14] Gerard Behague. Bossa and Bossas: recent changes in Brazilian urban popular music. *Ethnomusicology*, 17(2):209–233, 1973.

[15] Gerard Behague. Improvisation in Latin American musics. *Music Educators Journal*, 66(5):118–125, January 1980.

[16] Tolga Bektaş. Relationships between prosodic and musical meters in the Beste form of classical Turkish music. *Asian Music*, 36(1):1–26, Winter/Spring 2005.

[17] E. Bjorklund. A metric for measuring the evenness of timing system rep-rate patterns. SNS ASD Technical Note SNS-NOTE-CNTRL-100, Los Alamos National Laboratory, Los Alamos, U.S.A., 2003.

[18] E. Bjorklund. The theory of rep-rate pattern generation in the SNS timing system. SNS ASD Technical Note SNS-NOTE-CNTRL-99, Los Alamos National Laboratory, Los Alamos, U.S.A., 2003.

[19] John Blacking. Tonal organization in the music of two Venda initiation schools. *Ethnomusicology*, 14(1):1–56, 1970.

[20] Rose Brandel. The African hemiola style. *Ethnomusicology*, 3(3):106–117, September 1959.

[21] J. E. Bresenham. Algorithm for computer control of digital plotter. *IBM Systems Journal*, 4:25–30, 1965.

[22] Roy Brewer. The use of Habanera rhythm in rockabilly music. *American Music*, 17:300–317, Autumn 1999.

[23] Constantin Brăiloiu. Le rythme aksak. *Revue de Musicologie*, 33:71–108, 1951.

[24] Dave Brubeck. Unsquare Dance. In *Time Further Out*. Columbia Records, CS-8490 1961.

[25] Viggo Brun. Euclidean algorithms and musical theory. *Enseignement Mathématique*, 10:125–137, 1964.

[26] Mark J. Butler. Turning the beat around: Reinterpretation, metrical dissonance, and asymmetry in electronic dance music. *Music Theory Online*, 7(6), December 2001.

[27] Marc Chemillier. Ethnomusicology, ethnomathematics. The logic underlying orally transmitted artistic practices. In G. Assayag, H. G. Feichtinger, and J. F. Rodrigues, editors, *Mathematics and Music*, pages 161–183. Springer-Verlag, 2002.

[28] Marc Chemillier. Periodic musical sequences and Lyndon words. *Soft Computing*, 8:611–616, 2004.

[29] Marc Chemillier and Charlotte Truchet. Computation of words satisfying the "rhythmic oddity property" (after Simha Arom's works). *Information Processing Letters*, 86:255–261, 2003.

[30] John Miller Chernoff. *African Rhythm and African Sensibility*. The University of Chicago Press, Chicago, 1979.

[31] John Miller Chernoff. The rhythmic medium in African music. *New Literary History*, 22(4):1093–1102, Autumn 1991.

[32] Martin Clayton. *Time in Indian Music*. Oxford University Press, Inc., New York, 2000.

[33] J´erôme Cler. Pour une th´eorie de l'aksak. *Revue de Musicologie*, 80:181–210, 1994.

[34] J. Clough and J. Douthett. Maximally even sets. *Journal of Music Theory*, 35:93–173, 1991.

[35] John Clough, Nora Engebretsen, and Jonathan Kochavi. Scales, sets, and interval cycles: a taxonomy. *Music Theory Spectrum*, 21(1):74–104, Spring 1999.

[36] Justin Colannino, Francisco Gomez, and Godfried T. Toussaint. Steve Reich's Clapping Music and the Yoruba bell timeline. In Reza Sarhangi and John Sharp, editors, *Proceedings of BRIDGES: Mathematical Connections in Art, Music, and Science*, pages 49–58, London, United Kingdom, 2006.

[37] John Collins. *Highlife Time*. Anananses em Press, Ghana, 1996.

[38] Grosvenor Cooper and Leonard B. Meyer. *The Rhythmic Structure of Music*. The University of Chicago Press, Chicago, 1960.

[39] Aaron Copland. Jazz structure and influence. *Modern Music*, 4(2):9–14, 1927.

[40] Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein. *Introduction to Algorithms*. The MIT Press, Cambridge, Massachussetts, 2001.

[41] H. S. M. Coxeter. Music and mathematics. *The Canadian Music Journal*, VI:13–24, 1962.

[42] Larry Crook. A musical analysis of the Cuban rumba. *Latin American Music Review*, 3(1):92–123, Spring-Summer 1982.

[43] N. G. de Bruijn. Sequences of zeros and ones generated by special production rules. *Indagationes Mathematicae*, 43:27–37, 1981.

[44] N. G. de Bruijn. Updown generation of Beatty sequences. *Indagationes Mathematicae*, 51:385–407, 1989.

[45] Erik Demaine, Francisco Gomez-Martin, Henk Meijer, David Rappaport, Perouz Taslakian, Godfried Toussaint, Terry Winograd, and David Wood. The distance geometry of deep rhythms and scales. In *Proc. 17th Canadian Conf. Computational Geometry*, pages 160–163, University of Windsor, Windsor, Ontario, Canada, August 10-12 2005.

[46] Miguel D´ıaz-Bañez, Giovanna Farigu, Francisco G´omez, David Rappaport, and Godfried T. Toussaint. El comp´as flamenco: a phylogenetic analysis. In *Proceedings of BRIDGES: Mathematical Connections in Art, Music and Science*, pages 61–70, Southwestern College, Winfi eld, Kansas, July 30 - August 1 2004.

[47] Miguel D´ıaz-Bañez, Giovanna Farigu, Francisco G´omez, David Rappaport, and Godfried T. Toussaint. Similaridad y evoluci´on en la ritmica del flamenco: una uncursi´on de la matem´atica computational. *Gaceta de la Real Sociedad de Matematica Española*, 8(2):489–509, May-August 2005.

[48] Peter Dickinson. Improvisation: rhythm and texture. *The Musical Times*, 105(1458):612–614, August 1964.

[49] A. Dworsky and B. Sansby. *A Rhythmic Vocabulary*. Dancing Hands Music, Minnetonka, 1999.

[50] Issam El-Mallah and Kai Fikentscher. Some observations on the naming of musical instruments and on the rhythm in Oman. *Yearbook for Traditional Music*, 22:123–126, 1990.

[51] John Ellis, Frank Ruskey, Joe Sawada, and Jamie Simpson. Euclidean strings. *Theoretical Computer Science*, 301:321–340, 2003.

[52] Euclid. *Elements*. Dover, 1956. Translated by Sir Thomas L. Heath.

[53] Bob Evans. *Authentic Conga Rhythms*. Belwin Mills Publishing Corporation, Miami, 1966.

[54] Mary Farquharson. *Africa in America*. Discos Corazon, Mexico, 1992. [CD].

[55] Philip Franklin. The Euclidean algorithm. *The American Mathematical Monthly*, 63(9):663–664, November 1956.

[56] Jos´e Manuel Gamboa. *Cante por Cante: Discolibro Didactico de Flamenco*. New Atlantis Music, Alia Discos, Madrid, 2002.

[57] David Goldsworthy. Cyclic properties of Indonesian music. *Journal of Musicological Research*, 24:309–333, 2005.

[58] Kobi Hagoel. *The Art of Middle Eastern Rhythm*. OR-TAV Music Publications, Kfar Sava, Israel, 2003.

[59] Man-Young Hahn. The four musical types of Buddhist chant in Korea. *Yearbook for Traditional Music*, 15, East Asian Musics:45–58, 1983.

[60] Lennart Hallstrom. *African Drum Rhythms for Djembes, Bass Drums and Bells*. Lennart Hallstrom, Stockholm, 2000.

[61] Mitchell A. Harris and Edward M. Reingold. Line drawing, leap years, and Euclid. *ACM Computing Surveys*, 36(1):68–80, March 2004.

[62] Royal Hartigan, Abraham Adzenyah, and Freeman Donkor. *West African Rhythms for Drum Set*. Manhattan Music, Inc., 1995.

[63] Christopher F. Hasty. *Meter as Rhythm*. Oxford University Press, Oxford, England, 1997.

[64] Nathan Hesselink and Jonathan Christian Petty. Landscape and soundscape: Geomantic spatial mapping in Korean traditional music. *Journal of Musicological Research*, 23:265–288, 2004.

[65] Lee Hye-Ku. Quintuple meter in Korean instrumental music. *Asian Music*, 13(1):119–129, 1981.

[66] A. M. Jones. African rhythm. *Africa: Journal of the International African Institute*, 24(1):26–47, January 1954.

[67] A. M. Jones. *Studies in African Music*. Oxford University Press, Amen House, London, 1959.

[68] S. A. Floyd Jr. The sources and rescources of classic ragtime music. *Black Music Research Journal*, 4:22–59, 1984.

[69] S. A. Floyd Jr. Black music in the circum-Caribbean. *American Music*, 17(1):1–38, 1999.

[70] R. Kauffman. African rhythm: A reassessment. *Ethnomusicology*, 24(3):393–415, Sept. 1980.

[71] Michael Keith. *From Polychords to Pólya: Adventures in Musical Combinatorics*. Vinculum Press, Princeton, 1991.

[72] Reinhard Klette and Azriel Rosenfeld. Digital straightness - a review. *Discrete Applied Mathematics*, 139:197–230, 2004.

[73] Tőm Klőwer. *The Joy of Drumming: Drums and Percussion Instruments from Around the World*. Binkey Kok Publications, Diever, Holland, 1997.

[74] Roderic Knight. Mandinka drumming. *African Arts*, 7(4):24–35, Summer 1974.

[75] Donald E. Knuth. *The Art of Computer Programming*, volume 2, 3rd edition. Addison Wesley, Reading, Massachussets, 1998.

[76] Richard J. Krantz and Jack Douthett. Construction and interpretation of equal-tempered scales using frequency ratios, maximally even sets, and P-cycles. *Journal of the Acoustical Society of America*, 107(Pt. 1):2725–2734, May 2000.

[77] Gerhard Kubik. Oral notation of some West and Central African time-line patterns. *Review of Ethnology*, 3(22):169–176, 1972.

[78] Gerhard Kubik. Central Africa: An introduction. In Ruth M. Stone, editor, *The Garland Handbook of African Music*, pages 260–290. Garland Publishing, Inc., New York and London, 2000.

[79] Edward W. Large and Caroline Palmer. Perceiving temporal regularity in music. *Cognitive Science*, 26:1–37, 2002.

[80] David Locke. Principles of offbeat timing and cross-rhythm in Southern Ewe dance drumming. *Ethnomusicology*, 26:217–246, 1982.

[81] David Locke and Godwin Agbeli. Drum language in Adzogbo. *The Black Perspective in Music*, 9(1):25–50, Spring 1981.

[82] Wendell Logan. The ostinato idea in Black improvised music: A preliminary investigation. *The Black Perspective in Music*, 12(2):193–215, Autumn 1984.

[83] Justin London. Temporal asymmetries as period markers in isochronous and non-isochronous meters. In *7th International Conference on Music Perception and Cognition*, Sydney, Australia, July 2002.

[84] Justin London. *Hearing in Time: Psychological Aspects of Musical Meter*. Oxford University Press, Inc., New York, 2004.

[85] M. Lothaire. *Algebraic Combinatorics on Words*. Cambridge University Press, Cambridge, England, 2002.

[86] W. F. Lunnon and P. A. B. Pleasants. Characterization of two-distance sequences. *Journal of the Australian Mathematical Society (Series A)*, 53:198–218, 1992.

[87] Peter Manuel. The anticipated bass in Cuban popular music. *Latin American Music Review*, 6(2):249–261, Autumn-Winter 1985.

[88] Thomas J. Mathiesen. Rhythm and meter in ancient Greek music. *Music Theory Spectrum*, 7:159–180, Spring 1985.

[89] Alan P. Merriam. Flathead indian instruments and their music. *The Musical Quarterly*, 37:368–375, July 1951.

[90] Matthew Montfort. *Ancient Traditions–Future Possibilities: Rhythmic Training Through the Traditions of Africa, Bali and India*. Panoramic Press, Mill Valley, 1985.

[91] Robin Moore and Elizabeth Sayre. An Afro-Cuban Batá piece for Obatalá, king of the white cloth. In Michael Tenzer, editor, *Analytical Studies in World Music*, pages 120–160. Oxford University Press, New York, 2006.

[92] Sidney Moore. Thai songs in 7/4 meter. *Ethnomusicology*, 13(2):309–312, May 1969.

[93] Larry Morris. *Rhythm Catalog*. The Internet, http://www.drums.org/djembefaq, 2001.

[94] Robert Morris. Sets, scales, and rhythmic cycles: A classification of talas in Indian music. In *21st National Convention of the Society of Music Theory*, Chapel Hill, North Carolina, December 1998.

[95] Robert Morris. Architectonic composition in South Indian classical music. In Michael Tenzer, editor, *Analytical Studies in World Music*, pages 303–331. Oxford University Press, New York, 2006.

[96] Craig Morrison. *Go Cat Go: Rockabilly Music and Its Makers*. University of Illinois Press, Urbana, 1996.

[97] John P. Murphy. *Music in Brazil*. Oxford University Press, New York and Oxford, 2006.

[98] J. H. Nketia. *Drumming in Akan Communities of Ghana*. Thomas Nelson and Sons Ltd., Edinburgh, Scotland, 1963.

[99] J. H. Kwabena Nketia. *African Music in Ghana*. Northwestern Univ. Press, Britain, 1963.

[100] C. Stanley Ogilvy and John T. Anderson. *Excursions in Number Theory*. Oxford University Press, New York, 1966.

[101] Jeff Pressing. Cognitive isomorphisms between pitch and rhythm in world musics: West Africa, the Balkans and Western tonality. *Studies in Music*, 17:38–61, 1983.

[102] Vera Proca-Ciortea. On rhythm in Rumanian folk dance. *Yearbook of the International Folk Music Council*, 1:176–199, 1969.

[103] Putumayo. *Congo to Cuba*. Publisher, Address, 2002.

[104] Jose Luis Quintana and Chuck Silverman. *Changuito: A Master's Approach to Timbales*. Belwin-Mills Publishing Corp., Miami, 1998.

[105] Jay Rahn. Asymmetrical ostinatos in sub-saharan music: time, pitch, and cycles reconsidered. *In Theory Only*, 9(7):23–37, 1987.

[106] Jay Rahn. Turning the analysis around: African-derived rhythms and Europe-derived music theory. *Black Music Research Journal*, 16(1):71–89, 1996.

[107] E. M. Reingold and N. Dershowitz. *Calendrical Calculations: The Millenium Edition*. Cambridge University Press, Cambridge, UK, 2001.

[108] Timothy Rice. Aspects of Bulgarian musical thought. *Yearbook of the International Folk Music Council*, 12:43–66, 1980.

[109] Timothy Rice. *Music in Bulgaria*. Oxford University Press, Oxford, England, 2004.

[110] John Donald Robb. Rhythmic patterns of the Santo Domingo corn dance. *Ethnomusicology*, 8(2):154–160, May 1964.

[111] Olavo Al´en Rodr´ıguez. *Instrumentos de la Musica Folklórico-Popular de Cuba*. Centro de Investigaci´on y Desarrollo de la Musica Cubana, Havana, Cuba, 1997.

[112] Rene V. Rosalia. *Migrated Rhythm: The Tambú of Curaçao*. CaribSeek, 2002.

[113] Curt Sachs. *Rhythm and Tempo: A Study in Music History*. W. W. Norton, New York, 1953.

[114] Caroline Series. The geometry of Markoff numbers. *The Mathematical Intelligencer*, 7(3):20–29, 1985.

[115] Jeffrey O. Shallit. Pierce expansions and rules for the determination of leap years. *Fibonacci Quarterly*, 32(5):416–423, 1994.

[116] Patricia K. Shehan. Teaching music through Balkan folk dance. *Music Educators Journal*, 71(3):47–51, November 1984.

[117] Mirjana Simundza. Messiaen's rhythmical organization and classical Indian theory of rhythm (I). *International Review of the Aesthetics and Sociology of Music*, 18(1):117–144, June 1987.

[118] Mirjana Simundza. Messiaen's rhythmical organization and classical Indian theory of rhythm (II). *International Review of the Aesthetics and Sociology of Music*, 19(1):53–73, June 1988.

[119] Alice Singer. The metrical structure of Macedonian dance. *Ethnomusicology*, 18(3):379–404, September 1974.

[120] Doug Sole. *The Soul of Hand Drumming*. Mel Bay Productions Inc., Toronto, 1996.

[121] James A. Standifer. The Tuareg: their music and dances. *The Black Perspective in Music*, 16(1):45–62, Spring 1988.

[122] E. Thomas Stanford. The Mexican Son. *Yearbook of the International Folk Music Council*, 4:66–86, 1972.

[123] Ruth M. Stone. *Music in West Africa*. Oxford University Press, Oxford, England, 2005.

[124] Perouz Taslakian and Godfried T. Toussaint. Geometric properties of musical rhythm. In *Proceedings of the 16th Fall Workshop on Computational and Combinatorial Geometry*, Smith College, Northampton, Massachussetts, November 10-11 2006.

[125] Godfried T. Toussaint. A mathematical analysis of African, Brazilian, and Cuban *clave* rhythms. In *Proceedings of BRIDGES: Mathematical Connections in Art, Music and Science*, pages 157–168, Towson University, Towson, MD, July 27-29 2002.

[126] Godfried T. Toussaint. Classification and phylogenetic analysis of African ternary rhythm timelines. In *Proceedings of BRIDGES: Mathematical Connections in Art, Music and Science*, pages 25–36, Granada, Spain, July 23-27 2003.

[127] Godfried T. Toussaint. A mathematical measure of preference in African rhythm. In *Abstracts of Papers Presented to the American Mathematical Society*, volume 25, page 248, Phoenix, Arizona, January 7-10 2004. American Mathematical Society.

[128] Godfried T. Toussaint. The Euclidean algorithm generates traditional musical rhythms. In *Proc. of BRIDGES: Mathematical Connections in Art, Music and Science*, pages 47–56, Banff, Canada, July 31 - August 3 2005.

[129] Leo Treitler. Regarding meter and rhythm in the 'ars antiqua'. *The Musical Quarterly*, 65(4):524–558, October 1979.

[130] Ed Uribe. *The Essence of Brazilian Persussion and Drum Set*. CCP/Belwin Inc., Miami, Florida, 1993.

[131] Pedro van der Lee. Zarabanda: esquemas rítmicos de acompañamiento en 6/8. *Latin American Music Review*, 16(2):199–220, Autumn-Winter 1995.

[132] John Varney. *Colombian* Bambuco*: The Evolution of a National Music Style*. Griffith University, South Brisbane, Australia, 1999. Ph.D. Thesis.

[133] Bonnie C. Wade. *Thinking Musically*. Oxford University Press, Oxford, England, 2004.

[134] Marnix St. J. Wells. Rhythm and phrasing in Chinese tune-title lyrics; old eight-beat and its 3-2-3 meter. *Asian Music*, 23(1):119–183, (Autumn, 1991 - Winter, 1992).

[135] Caroline Card Wendt. Tuareg music. In Ruth M. Stone, editor, *The Garland Handbook of African Music*, pages 206–227. Garland Publishing, Inc., New York and London, 2000.

[136] O. Wright. *The Modal System of Arab and Persian Music AD 1250-1300*. Oxford University Press, Oxford, England, 1978.

[137] Luis Felipe Ramón y Rivera. Rhythmic and melodic elements in Negro music of Venezuela. *Yearbook of the International Folk Music Council*, 14:56–60, 1962.

[138] Maury Yeston. *The Stratification of Musical Rhythm*. Yale University Press, New Haven, CT., 1976.

[139] Paul F. Zweifel. Generalized diatonic and pentatonic scales: a group-theoretic approach. *Perspectives of New Music*, 34(1):140–161, Winter 1996.