Geometry, to appear in 1991.

[To92] Toussaint, G. T., ed., *Proceedings of the IEEE*, Special Issue on Computational Geometry, to appear in 1992.

[YKII88] Yamamoto, P., Kato, K., Imai, K. and Imai, H., “Algorithms for vertical and orthogonal linear L_1 approximation of points,” *Proc. of the ACM Symp. on Computational Geometry*, Urbana-Champaign, 1988, pp. 352-361.

on Computational Geometry, Berkeley, 1990, pp. 211-215.

[Sh78] Shamos, M. I., "Computational geometry," Ph.D. thesis, Yale University, May 1978.

[SH75] Shamos, M. I. and Hoey, D., "Closest-point problems," *16th Annual IEEE Symposium on Foundations of Computer Science*, October 1975, pp. 151-162.

[SSH87] Schwartz, J. T., Sharir, M., and Hopcroft, J., *Planning, Geometry, and the Complexity of Robot Motion,* Norwood, 1987.

[St1882] Steiner, J., *Gesammelte Werke*, 2. Bd. Berlin 1882, p. 45.

[St1884] Sturm, R., "Bemerkungen und Zusatze zu Steiners Aufsazen uber Maxima und Minima," *Journal fur reine und angewande Mathematik*, vol. 96,1884, pp.36-77.

[SY87] Schwartz, J. T. and Yap, C. K., *Algorithmic and Geometric Aspects of Robotics*, Erlbaum, 1987.

[Sy1857] Sylvester, J. J., "A question in the geometry of situation," *Quarterly Journal of Mathematics*, vol. 1, 1957, p. 79.

[To83a] Toussaint, G. T., "Solving geometric problems with the rotating calipers," *Proc. of IEEE MELECON*, Athens, 1983.

[To83b] Toussaint, G. T., "Computing largest empty circles with location constraints," *International Journal of Computer and Information Sciences,* vol. 12, October 1983, pp. 347-358.

[To86] Toussaint, G. T., "An Optimal Algorithm for Computing the Relative Convex Hull of a Set of Points in a Polygon, "*Signal Processing III: Theories and Application*s, Proc. of EURASIP-86, Part 2, The Hague, September 1986.

[To85a] Toussaint, G. T., "A historical note on convex hull finding algorithms," *Pattern Recognition Letters*, vol. 3, January 1985, pp. 21-28.

[To85b] Toussaint, G. T., ed., *Computational Geometry,* North-Holland, 1985.

[To88a] Toussaint, G. T., ed., *Computational Morphology*, North-Holland, 1988.

[To88b] Toussaint, G. T., ed., *The Visual Computer*, Special Issue on Computational Geometry, vol. 3, No. 6, May 1988.

[To89] Toussaint, G. T., "Computing geodesic properties of polygons," *Revue D'Intelligence Artificielle*, vol. 3, 1989, pp. 9-42.

[To90b] Toussaint, G. T., *Workshop on Illuminating Sets*, Bellairs Research Inst. of McGill Univ., Feb. 1990.

[To90a] Toussaint, G. T., "An output-complexity-sensitive polygon triangulation algorithm," *Proc. Computer Graphics International'90,* Singapore, June 1990, pp. 443-466.

[To91] Toussaint, G. T., ed., *Pattern Recognition Letters*, Special Issue on Computational *rithmica*, vol. 1, pp. 193-211

[Me83] Megiddo, N., "Linear time algorithm for linear programming in R^3 and related problems," *SIAM Journal of Computing*, vol. 12, November 1983, pp. 759-776.

[Me84] Megiddo, M., "Linear programming in linear time when the dimension is fixed," *Journal of the Association for Computing Machinery*, vol. 31, 1984, pp. 114-127.

[Me89] Megiddo, M., "On the ball spanned by balls," *Discrete and Computational Geometry*, vol. 4, 1989, pp. 605-610.

[Meh84] Mehlhorn, K., *Multidimensional Searching and Computational Geometry*, Springer-Verlag, 1984.

[MN80] Morris, J. G. and Norback, J. P., "A simple approach to linear facility location," *Transportation Science*, vol. 14, 1980, pp. 1-8.

[MS90] Melissaratos, E. A. and Souvaine, D. L., "On Solving Geometric Optimization Problems Using Shortest Paths," *Proc. of the ACM Symp. on Computational. Geometry*, Berkeley, 1990, pp. 350-359.

[Ni81] Niven, I., *Maxima and Minima Without Calculus*, The Mathematical Association of America, 1981, p. 163.

[O83] O'Rourke, J., "An on-line algorithm for fitting lines between data ranges," *Communication of the ACM*, 24, 1981, pp. 574-578.

[O'R87] O'Rourke, J., *Art Gallery Theorems and Algorithms*, Oxford University Press, 1987.

[Pr77] Preparata, F. P., "Minimum spanning circle," in *Steps into Computational Geometry*, F. P. Preparata, ed., University of Illinois, March 1977, pp. 3-5.

[Pr83] Preparata, F., ed., *Computational Geometry*, JAI Press, 1983.

[PS85] Preparata, F. P. and Shamos, M. I., *Computational Geometry*, Springer-Verlag, 1985.

[PS86] Pollack, R., Sharir, M., "Computing the geodesic center of a simple polygon," in *Proc. Workshop on Movable Separability of Sets*, G. Toussaint, *ed*., Bellairs Research Institute, Barbados, February 1986.

[PSR89] Pollack, R., Sharir, M., and Rote, G., "Computing the geodesic center of a simple polygon," *J. of Discrete and Comp. Geometry*, Vol.4, No. 6, 1989, pp. 611-626.

[RT57] Rademacher, H. and Toeplitz, O., *The Enjoyment of Mathematics*, Princeton University Press, 1957.

[Sc1890] Schwarz, H. A., G*esammelte Mathematische Abhandlungen, 2*. Bd., Berlin 1890, pp. 344-345.

[Se90] Seidel, R., "Linear programming and convex hulls made easy," *Proc. of the ACM Symp.*

tures, Lecture Notes in Computer Science 382, F. Dehne, J.-R. Sack and N. Santoro eds., Springer-Verlag, 1989, pp. 183-191.

[HR88] Houle, M. E. and Robert J.-M., "Orthogonal weighted linear approximation and applications," Tech. Rept. SOCS-88.13, McGill University, August 1988.

[HS86] Hart, and Sharir, M., "Nonlinearity of Davenport-Schintzel sequences and of generalized path compression schemes," *Combinatorica*, 6, 1986, pp.151-177.[H]Hershberger, J., "Finding the upper envelope of n line segments in $O(n \log n)$ time," *Information Processing Letters*, vol. 33, 1989, pp. 169-184.

[HS] Hershberger, J. and Suri, S., "Off-line maintenance of planar configurations," *SODA*, San Francisco, January 1991.

[HT88] Houle, M. E. and Toussaint, G. T., "Computing the width of a set," *IEEE Transactions in Pattern Analysis and Machine Intelligence*, vol. 10, 1988, pp. 761-765.

[HT85] Hildebrandt, S. and Tromba, A., *Mathematics and Optimal Form*, Scientific American Books, Inc., 1985, p. 60.

[IKY89] Imai, H., Kato, K. and Yamamoto, P., "A linear-time algorithm for linear L_1 approximation of points," *Algorithmica*, vol. 4, 1989, pp. 77-96.

[KET90] Kong, X., Everett, H. and Toussaint, G. T., "Graham scan triangulates a simple polygon," *Pattern Recognition Letters*, in press.

[KK85] Karzakis, J. and Karagiorgis, P., "A method to locate the maximum circle inscribed in a polygon, "*Belgian Journal of Operations Research*, *Statistics and Computer Science*," vol. 26, No. 3. 1985, pp. 3-36.

[Kl89] Klein, R., *Concrete and Abstract Voronoi Diagrams*, Springer-Verlag, 1989.

[KR85] Klotzler, R. and Rudolph, H., "Zur analytischen und algorithmischen Behandlung eines geometrischen Optimierungsproblems von J. Steiner," *Optimzation*, vol. 16, 1985, pp. 833-848.

[LD81] Lee, D. T. and Drysdale III, R. L., "Generalizations of Voronoi diagrams in the plane," *SIAM Journal on Computing*, vol. 10, 1981, pp. 73-87.

[Le80] Lee, D. T., "Farthest neighbor Voronoi diagrams and applications," Tech. Rept. #80-11-FC-04, Northwestern University, November 1980.

[Le82] Lee, D. T., "Medial axis transformation of a planar shape," *IEEE Transactions on Pattern Analysis and Machine Intelligence*, vol. PAMI-4, July 1982, pp. 363-369.

[LP84] Lee, D. T., and Preparata, F. P., "Euclidean shortest paths in the presence of rectilinear barriers," *Networks,* Vol. 14, No. 3., 1984, pp. 393-410.

[LS87] Leven, D. and Sharir, M., "Intersection and proximity problems and Voronoi diagrams," in *Algorithmic and Geometric Aspects of Robotics Volume 1,* J. T. Schwartz and C.-K. Yap eds., Lawrence Ertlbaum Associates, 1987, pp. 187-228.

[LW86] Lee, D. T. and Wu, Y. F., "Geometric complexity of some location problems," *Algo-*

1989.

[CN86] Chin, W. and Ntafos, S., “Optimum watchman routes,” *Proc. of the ACM Symp. on Computational Geometry*, Yorktown Heights, 1986, 24-33.

[CR41] Courant, R. and Robbins, H., *What is Mathematics?*, Oxford Univ. Press, 1941, 346-352.

[DT81] Devroye, L. and Toussaint, G. T., “A note on linear expected time algorithms for finding convex hulls,” *Computing*, vol. 26, pp. 361-366

[Dy86] Dyer, M. E., “On a multidimensional search technique and its applications to the Euclidean one-center problem,” *SIAM Journal on Computing*, vol. 15, 1986, pp. 725-738.

[Ed85] Edelsbrunner, H., “Finding transversals for sets of simple geometric figures,” *Theoretical Computer Sciences*, vol. 35, 1985, pp. 55-69.

[Ed87] Edelsbrunner, H., *Algorithms in Combinatorial Geometry*, Springer-Verlag, 1987.

[EG89] Edelsbrunner, H. and Guibas, L. J., “Topologically sweeping an arrangement,” *Journal of Computer and System Sciences*, vol 38, 1989, pp. 165-194.

[EH72] Elzinga, J. and Hearn, D. W., “Geometrical solutions for some minimax location problems,” *Transportation Science*, vol. 6, 1972, pp. 379-394.

[EMPRWW82] Edelsbrunner, H., Maurer, H. A., Preparata, F. P., Rosenberg, A. L., Welzl, E. and Wood, D., “Stabbing line segments,” *BIT*, vol. 22, 1982, pp. 274-281.

[ET90] Everett, H. and Toussaint, G. T., “Illuminating objects in the plane with point light sources,” *Proc. Third Australasian Workshop on Combinatorial Algorithms*, Ubud, Indonesia, June 11-15, 1990.

[EW89] Egyed, P. and Wenger, R., “Stabbing pairwise disjoint translates in linear time,” *Proc. of the ACM Symp. on Computational Geometry*, Saarbruchen, 1989, pp. 364-369.

[Fo86] Focke, J., “A finite descent method for STEINER’s problem of inpolygons with minimal circumference,” *Optimization*, 17, 1986, pp. 355-366.

[FM84] Fournier, A. and Montuno, D. Y., “Triangulating Simple Polygons and Equivalent Problems,” *ACM Transactions on Graphics*, 1984, 153-74.

[FW74] Francis, R. L. and White, J. A., *Facility Layout and Location*: *An Analytical Approach*, Prentice-Hall, Inc., 1974.

[GSLHT87] Guibas, L., Hershberger, J., Leven, D., Sharir, M. and Tarjan, R. E., “Linear Time Algorithms for Visibility and Shortest Path Problems Inside Triangulated Simple Polygons,” *Algorithmica*, vol. 2, 1987, 209-233.

[He89] Hershberger, J., “Finding the upper envelope of n lines segments in $O(n \log n)$ time,” *Information Processing Letters*, vol. 33, 1989, pp. 169-174.

[HIIR89] Houle, M. E., Imai, H., Imai, K. and Robert J.-M., “Weighted orthogonal linear L_∞-approximation and applications,” *Proc. of the Workshop on Algorithms and Data Struc-*

65-79.

[AT78] Akl, S. G. and Toussaint, G. T., "Efficient convex hull algorithms for pattern recognition applications," *Proc. Fourth International Joint Conf. on Pattern Recognition*, Kyoto, Japan, 1978.

[AT81] Avis, D. and Toussaint, G. T., "An efficient algorithm for decomposing a simple polygon into star-shaped pieces," *Pattern Recognition*, vol. 13, 1981, pp. 295-298.

[Ba84] Bajaj, C., "Geometric optimization and computational complexity," Ph.D. thesis, Tech. Rept. TR-84-629, Cornell University, 1984.

[BCEKSTU90] Bhattacharya, B. K., Czyzowics, J., Egyed, P., Keil. M., Stojmenovic, I., Toussaint, G. T. and Urrutia, J., "Computing shortest transversals of sets," manuscript in preparation.

[BE86] Bhattacharya, B. K. and ElGindy H., "An efficient algorithm for an intersection problem and an application," Tech. Rept. No. 86-25, Dept. of Computer and Information Science, University of Pennsylvania, April 1986.

[BS67] Bass, L. J. and Schubert, S. R., "On finding the disc of minimum radius containing a given set of points," *Mathematics of Computation*, vol. 21, 1967, pp. 712-714.

[BT83] Bhattacharya, B. K. and Toussaint, G. T., "Time-and-storage-efficient implementation of an optimal planar convex hull algorithm," *Image and Vision Computing*, vol. 1, no. 3, August 1983, pp. 140-144.

[BT85] Bhattacharya, B. K. and Toussaint, G. T., "On geometric algorithms that use the furthest-point Voronoi diagram," in *Computational Geometry*, ed., G. T. Toussaint, North-Holland, 1985, pp. 43-61.

[BT90] Bhattacharya, B. K. and Toussaint, G. T., "Computing shortest transversals," *Computing*, in press.

[BCEKSTU] Bhattacharya, B., Czyzowics, J., Egyed, P., Stojmenovic, I., Toussaint, G. T., Urrutia, J., "Computing shortest transversals," manuscript in preparation.

[CEERSSTU90] Czyzowics, J., Egyed, P., Everett, H., Rappaport, D., Shermer, T., Souvaine, D., Toussaint, G. T. and Urrutia, J., "The aquarium keeper's problem," *SODA*, San Francisco, January 1991.

[Ch82] Chazelle, B., "A theorem on polygon cutting with applications," *Proc. 23rd IEEE Symp. on Foundations of Computer Science*, Chicago, Nov. 1982, pp. 339-349.

[Ch90] Chazelle, B., "Triangulating a simple polygon in linear time," Technical Report CS-TR-264-90, Princeton Univ., May 1990.

[Ch75] Chvatal, V., "A combinatorial theorem in plane geometry," *Journal of Combinatorial Theory, Series B*, vol. 18, 1975, pp. 39-41.

[CN89] Chin, W. and Ntafos, S., "Optimum zoo-keeper routes," *Congressus Numerantium*,

tion principle and shortest-path maps. They also generalize their method by using *relative convex hulls* [To86] to provide a linear-time algorithm for polygons which are not convex. Chin and Ntafos [CN86] have used a somewhat similar combination of the reflection principle and shortest-paths to solve a related problem of computing shortest watchman routes in rectilinear polygons.

The Aquarium Keeper's Problem is related to another problem considered by Chin and Ntafos [CN89], the *Zoo Keeper's Problem*. In the Zoo Keeper's Problem, given a simple polygon P of n vertices, k convex polygons (cages) attached to edges of P, and an entry point x on the boundary of P, the goal is to find the minimum perimeter tour in P and not in the interior of any cage, starting and ending at x, that visits every cage. Their paper contains an $O(n \log^k n)$ time algorithm and refers to an $O(n^2)$ algorithm for the problem as well. If we consider the cages as edges of P, *i.e.,* the edges of P represent the front glass plates of a series of aquariums and have n cages, then the Zoo Keeper's Problem reduces to a simplified version of the Aquarium Problem in which a fixed starting point is given. Note that even without this restriction, the complexity of the algorithm in [CEERSSTU90] is appreciably less than either $O(n \log^k n)$ or $O(n^2)$.

6. Multiple Facility Location of the Visibility Kind

Consider a large warehouse in which it is desired to install a surveillance system consisting of a set of cameras. The cameras are to be installed in fixed positions but are allowed to rotate through 360°. It is required that every portion of the warehouse fall in the field of view of at least one camera as it makes a full revolution. On the other hand we don't want to install too many cameras to do the job. This is an example of a *multiple-facility* location problem of the *visibility* kind where the facilities are points (camera locations) and the warehouse is the "customer." We can model the warehouse as a simple polygon of n vertices in the plane. A well known theorem that concerns this type of multiple-facility location problem is Chvatal's *Art-Gallery Theorem* [O'R87]. This theorem states that $n/3$ cameras are always sufficient and sometimes necessary to do the job [Ch75]. Avis and Toussaint [AT81] presented an $O(n \log n)$ time algorithm for finding the location of these cameras. The book by O'Rourke [O'R87] is devoted entirely to these types of facility location problems for simple polygons as customers. For the case of simple objects such as circles and squared as customers a survey of results can be found in [ET90].

7. References

[AB87] Atallah, M. and Bajaj, C., "Efficient algorithms for common transversals," *Information Processing Letters*, vol. 25, 1987, pp. 87-91.

[ARW89] Avis, D., Robert, J.-M. and Wenger, R., "Lower bounds for line stabbing," *Information Processing Letters*, vol. 33, 1989, pp. 59-62.

[AT85] Asano, T. and Toussaint, G. T., "Computing the geodesic center of a simple polygon," Technical Report SOCS-85.32, McGill University, 1985.

[AT86] Asano, T. and Toussaint, G. T., "Computing the geodesic center of a simple polygon," in *Perspectives in Computing: Discrete Algorithms and Complexity, Proc. of Japan-US Joint Seminar,* D. S. Johnson, A. Nozaki, T. Nishizeki, H. Willis, *eds.*, June 1986, pp.

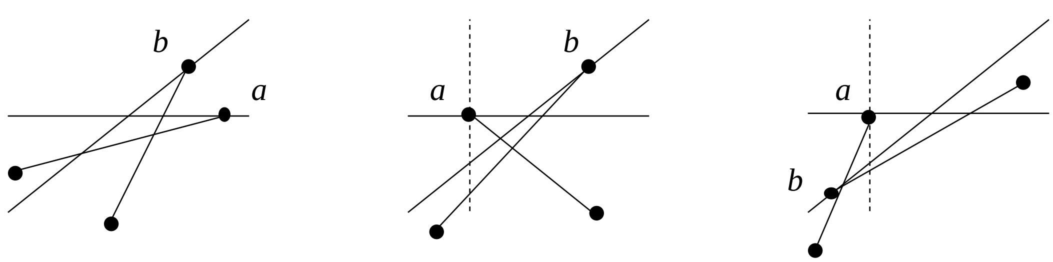
calipers. Finally, if the line segments are already sorted, the running time of the algorithm is $O(n)$.

5. Polygonal Facilities

Just as we may have a point facility in a polygonal customer [KK85] we may also have a polygonal facility with different types of customers. The polygonal facility can be viewed as a circuit of railway lines for example or a circuit of power lines to provide electricity to a polygonal region, etc. When the customers are points then we of course obtain the old and famous *travelling salesman problem* which is well known and hence we do not consider here. Much more novel is the situation where the customer is itself a simple polygon or a set of convex polygons lying in the interior of a simple polygon and we briefly touch upon this topic below.

Although variously attributed to Fagnano [Ni81] and Steiner [Fo86], [KR85], [St1884], [St1882], many sources insist that, over one hundred years ago, Schwarz [Sc1890] not only solved the problem of computing the minimum perimeter triangle with one vertex on each edge of a given triangle but that he also posed the problem. In any case, there is consensus that Schwarz used the *reflection principle* to show that the foot points of the altitudes of an acute triangle are the vertices of the minimum inscribed polygon. For obtuse triangles, the minimum perimeter inscribed triangle is realized by twice the shortest altitude, *i.e.*, it is degenerate with two vertices coinciding with the obtuse vertex of the input triangle. In 1985, Klotzler and Rudolph [KR85] used a semi-infinite simplex-method to determine the minimum perimeter polygon with one vertex on each edge of a given convex polygon, or the minimum perimeter *inpolygon*. In 1986, Focke [Fo86] presented an algorithm for computing the optimum inpolygon, using a *finite descent* method resulting from Schwarz' reflection principle and coordinate-wise descent. He presents four examples to illustrate efficient performance of the algorithm, but provides no formal complexity analysis. In addition, his classification of correct solutions omits one realizable type and it is unclear whether his algorithm would in fact detect the correct solution.

At a recent workshop, Toussaint [To90b] posed the problem somewhat differently: he asked for the shortest closed path inside a simple polygon (*Aquarium Keeper's Tour*) which visits every edge at least once. If the polygon is convex, then the optimum path is, in fact, the minimum perimeter inpolygon, and [CEERSSTU90] present a linear-time algorithm which uses the reflec-



(a) *aba* impossible (b) *abab* impossible (c) *ababa* impossible

Fig. 5

the list $\theta_0 = 0°, \theta_1, \theta_2..., \theta_{s\text{-}1}, \theta_s = 90°$.

Step 3: Set $i = 0$.

Step 4: Compute $CH(UP(F, \theta_i))$ and $CH(LH(F, \theta_i)$.

Step 5: Find the thinnest strip transversal with orientation θ s. t. $\theta_i \leq \theta \leq \theta_{i+1}$.

Step 6: Update $CH(UP(F, \theta_i))$ and $CH(LH(F, \theta_i)$ and produce $CH(UP(F, \theta_{i+1}))$ and $CH(LH(F, \theta_{i+1})$.

Step 7: Let $i = i + 1$. If $i < s$, go to 5.

Step 8: Output the thinnest strip overall the strips found in Step 5.

The crucial point of this algorithm for its running time is the convex hull updating in Step 6. By using the recent algorithm by Hershberger and Suri [HS91], it is possible to do the $O(n)$ insertions and deletions in $O(\log n)$ amortized time per operation when all the points to be processed are known in advance. Then, since Step 1 can be done in $O(n)$ time using the rotating calipers algorithm, as mentioned before, and Steps 2 and 3 and all the convex hull updates in $O(n \log n)$ time, the algorithm runs in $O(n \log n + p)$ time where p is the number of vertices visited by the rotating calipers algorithm in Step 5.

An upper bound on p can be derived from the theory of Davenport-Schintzel sequences. The sequence of all vertices visited in Step 5 by the lower tangents form a sequence of labels such that no two adjacent symbols are the same and, furthermore, it is impossible to have the subsequence *a...b...a...b...a*. Fig. 5 illustrates all the different cases. In case 3(c), for example, it is possible to have the sequence *a...b...a...b*. To obtain the sequence *a...b...a...b*, the tangent has to have an orientation greater than that of the edge containing the vertex *a*. So, the vertex *a* is replaced by the other endpoint and it will never be looked at again. Therefore the sequence *a...b...a...b...a* will never appear. Thus, the sequence of vertices corresponds to a $(n, 3)$-Davenport-Schintzel sequence and the $O(n\,\alpha(n))$ upper bound of Hart and Sharir [HS86], where $\alpha(n)$ is the functional inverse of Ackermann's function, gives an upper bound on p. Therefore, the overall running time of the algorithm is $O(n \log n)$. For vertical line segments, the algorithm consists of only one phase of computing the convex hulls of the tangent points and finding the thinnest strip transversal with the rotating

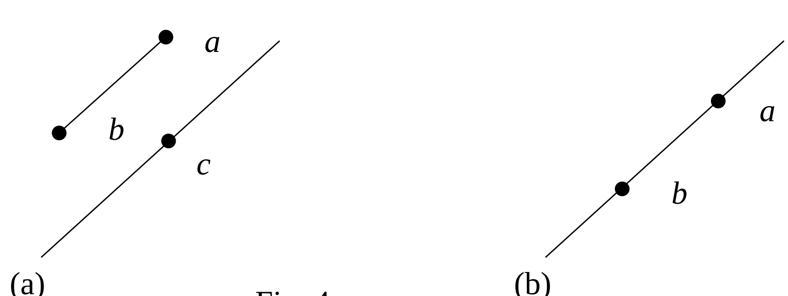


Fig. 4

orientation between 0° and 90° only. To solve the unrestricted problem, the algorithm should be used twice.

The algorithm can be divided into two parts: finding the thinnest strip between two consecutive critical directions and updating the information needed to find the strip transversal. Since the tangent points do not change between two adjacent critical directions, say θ_1 and θ_2, the thinnest strip transversal can be compute easily in that range. When the convex hulls of the upper tangent points $CH(UP(F, \theta_1))$ and the lower tangent points $CH(LP(F, \theta_1))$ have been computed, the thinnest strip transversal can be found by determining the lower tangent $t_l(CH(UP(F, \theta_1), \theta_1))$ and the upper tangent $t_u(CH(LP(F, \theta_1), \theta_1))$ and rotating them until they have an orientation θ_2 [HT88]. Note that if, at any moment, the lower tangent becomes "above" or "on" the upper tangent, a line transversal (i.e. a strip with width 0) is found. Finally, the extreme case where the tangents have orientation θ_2 should be looked at carefully. Even if $UP(F, \theta_1) \neq UP(F, \theta_2)$, the lower tangents $t_l(CH(UP(F, \theta_1), \theta_2)$ and $t_l(CH(UP(F, \theta_2), \theta_2)$ are the same since $UP(F, \theta_2) = (UP(F, \theta_1)) \setminus \{a\}) \cup \{b\}$, where $[a, b]$ is an edge of one member of F and has an orientation θ_2 (if many edges are parallel more than one point can be replaced at the same time). Fig. 4 illustrates the two cases which can occur. The segment $[a, b]$ either belongs to the lower tangent or it does not. In either case, replacing the point a by b does not affect the tangent. Furthermore, this gives a simple way to update $CH(UP(F, \theta_1))$ to obtain $CH(UP(F, \theta_2))$. One point should be added and one should be deleted from the convex hull. Naturally, theses observations also hold for $CH(LP(F, \theta_2))$ and $t_u(CH(LP(F, \theta_2), \theta_2))$.

From the above facts, the algorithm is simply a succession of subproblems consisting of finding the thinnest strip transversal in a given orientation range and updating all the information to be able to process efficiently the next range. The algorithm can now be stated as follows:

Strip Transversal Algorithm:

Step 1: Compute the sequence of tangent points for each member of F.

Step 2: Compute the critical directions and sort them according to their slopes to produce

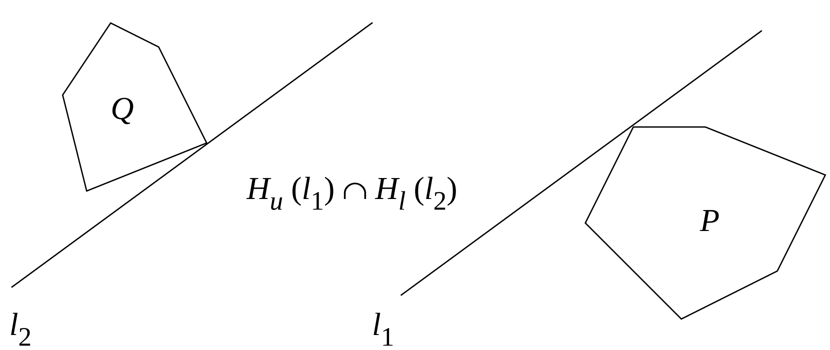


Fig. 3 Illustrating the proof of Lemma 1.

that intersects a set of line segments. The *medial axis* of such a strip is the desired line facility.

In this section we present an $O(n \log n)$ time solution to the thinnest strip transversal problem for a set of polygonal customers with a total of n vertices. Since the line transversal problem can be reduced to the strip transversal problem, the $\Omega(n \log n)$ time lower bound applies here as well. Therefore, the algorithm given is optimal. The algorithm depends heavily on the use of the *rotating calipers* paradigm [To83a] and the dynamic maintenance of convex hulls. The next sub-section presents the geometrical facts needed to prove the correctness of the algorithm and the last sub-section presents the algorithm and its running time analysis.

4.2.1 Geometric Preliminaries

The line l, defined by the equation $h_1x + h_2y + h_3 = 0$ ($h_2 \geq 0$ or $h_2 = 0$ and $h_1 = -1$), determines the upper half-plane $H_u(l) = \{p \mid h_1x_p + h_2y_p + h_3 \geq 0\}$ and the lower half-plane $H_l(l) = \{p \mid h_1x_p + h_2y_p + h_3 \leq 0\}$. A strip S is defined as the closed region bounded by two parallel lines l_1 and l_2. More formally, S is given by $H_u(l_1) \cap H_l(l_2)$. Therefore, l_1 represents the "lower boundary" and l_2 the "upper boundary" of S. The *width* of S is the orthogonal distance between l_1 and l_2 and the *orientation* of S is the angle l_1 makes with the positive x-axis. For a given orientation θ, the *lower tangent* of a polygon P is the line $t_l(P, \theta)$ with orientation θ such that $t_l(P, \theta) \cap P \neq \emptyset$ and $P \subset H_u(t_l(P, \theta))$. The *lower point* $p_l(P, \theta)$ of polygon P is any point in $t_l(P, \theta) \cap P$ and $LP(F, \theta)$ is the set of all lower points of members of F. Finally, the *upper tangent* $t_u(P, \theta)$, the *upper point* $p_u(P, \theta)$ and the set of all upper points $UP(F, \theta)$ are defined similarly.

The following lemma gives a simple criterion for determining if a strip with a given orientation is optimal and can readily be proven (see Fig. 3).

Lemma 1: For a given orientation θ, a strip S of width greater than 0 is the thinnest possible for a family F if, and only if, there exist $P, Q \in F$ such that $P \cap H_u(l_1) \subseteq l_1$ and $Q \cap H_l(l_2) \subseteq l_2$.

An equivalent way to formulate Lemma 1 is to say that $l_1 = t_l(CH(UP(F, \theta)), \theta)$ (where $CH(X)$ represents the convex hull of X) i.e. the lower tangent of the upper points of F and $l_2 = t_u(CH(LP(F, \theta)), \theta)$ i.e. the upper tangent of the lower points of F.

Since Lemma 1 implies that only lower and upper points are necessary, it is important to determine these two sets efficiently. For a convex polygon p with k vertices, the sequence of lower and upper points can be determined easily using the rotating calipers algorithm [To83a]. By starting with horizontal parallel lower and upper tangents and rotating them in a counterclockwise direction until the tangents become vertical, the sequences of $p_l(P, \theta)$ and $p_u(P, \theta)$ for θ between 0° and 90° can be computed in $O(k)$ time. When one of the tangents contains an edge of a polygon, the orientation of the tangents is called a *critical direction*. In such a case, the choice of the tangent point can be ambiguous. Nevertheless it is possible to overcome this problem by choosing the vertex which will be the next tangent point as the tangents rotate. In this way, the sequences of tangent points are well defined and change only at critical directions.

4.2.2 Strip Transversal Algorithm

For simplicity, the algorithm presented here computes the thinnest strip transversal with presented in the past. Houle and Toussaint [HT88] solved the unweighted case, Edelsbrunner [Ed85] claimed to be able to solve the weighted case optimally and, finally, Houle *et al.* [HIIR89] also presented an optimal solution for the weighted case which can be extended to higher dimensions. For the L_1-norm, Houle and Robert [HR88] and Yamamoto *et al.* [YKII88] presented two different $O(n^2)$ time algorithms both based on performing a topological sweep of an arrangement of hyperplanes [EG89]. The first algorithm can be used to solve the problem in higher dimensions and the second can be transformed to solve the unweighted case in $O(n^{1.5} \log^2 n)$ time.

4.2 Simple convex objects as customers

The line-facility location problem for simple-convex-customers has been solved in a very narrow setting in the past. It corresponds to the *line transversal problem* investigated in both the mathematics [DGK63], [HDK64] and computer science [AB87], [Ed85], [EW89] literatures. Let S be a family of convex sets. The line transversal problem consists of finding a line which intersects each member of S if it exists.

For a family of n simple convex objects, by combining the algorithms of Atallah and Bajaj [AB87] and Hershberger [He89], it is possible to find a line transversal in $O(n \log n)$ time. Recently, Avis, Robert and Wenger [ARW90] proved that such an algorithm is optimal. When the objects are restricted to be mutually disjoint translates of a convex set, Egyed and Wenger [EW89] gave an optimal linear time algorithm to solve the problem. For line segments, Edelsbrunner and Guibas [EG89] gave an $O(n^2)$ time algorithm to find a line which intersects the maximum number of them by sweeping the line arrangement in the dual space. The same method will work for a family of convex polygons with a fixed number of vertices.

Recently, Bhattacharya and Toussaint [BT90] introduced the problem of computing the shortest line segment transversal for a family of line segments. This problem can be interpreted as finding the smallest line-segment-facility which intersects each line-segment-customer. They proposed a $O(n \log n)$ time algorithm to find such a transversal for a set of non-intersecting line segments. They also gave a $O(n \log^2 n)$ time algorithm for a set of lines or, possibly intersecting, line segments. Bhattacharya *et al.* [BCEKSTU90] improved this result and obtained an optimal $O(n \log n)$ time algorithm for a family of line segments. They also presented an $O(n \log n)$ time algorithm for a set of lines, but the optimality of this algorithm has not yet been determined. It should be pointed out here that if it is desired to find the shortest *vertical* line segment facility which intersects each of n line-customers then the problem is identical to the point-facility for lines as customers using the Euclidean *vertical* distance. As pointed out in sub-section 3.2 this problem can be expressed as a linear program in three variables and can thus be solved in O(n) time.

The line-facility location for simple-object-customers appears not to have been examined before. For example, suppose we have n line segments which represent the customers, one possible problem is to find the line which minimizes the maximum Euclidean orthogonal distance between a line segment and the line. Equivalently, the problem can be stated as finding the thinnest strip only the two following facts:

i) $FPVD(S)$ consists of at most a linear number of edges and vertices,

ii) each edge is a connected subset of a bisector between two sites (i.e. the locus of points which are equidistant from two sites).

An optimal solution of the point-facility location problem must belong to the *FPVD* of the customers since at least two customers must be externally tangent to the circle corresponding to the solution. Therefore, for mutually disjoint line-segment-customers, Fortune's algorithm [Fo87] can compute the *FPVD* in $O(n \log n)$ time and the solution can then be found in linear time by examining the vertices and the "edges" each of which is composed of at most seven straight line segments and parabolic arcs. For more complex customers, we have to be more careful. If the customer sites have to many vertices, the edges of the *FPVD* can be formed by too many segments and the last part of the algorithm will take more than linear time. If we restrict the customers to be small convex polygons with a fixed upper bound on the number of vertices, the algorithm will still find the solution in linear time from the *FPVD*. However, it is no longer possible to compute the *FPVD* in $O(n \log n)$ time and we have to resort to Lee and Drysdale's algorithm [LD81] which runs in $O(n \log^2 n)$ time. The special case of circle-customers can be solved easily using linear programming. Megiddo [Me89] presented a linear time algorithm to solve this problem in any fixed dimension.

4. Line-Facility Location

The general formulation of the last section still applies in this case. The only difference will be in the definition of the distance function between the customers and the line-facility.

4.1 Points as customers

The line-facility location problem for point-customers is simply the well known *line-fitting* or *linear approximation problem*. Many solutions have been presented over the years for the different norms and distance metrics used.

For the Euclidean *vertical* distance, it is possible to solve the problem optimally for the L_1, L_2 (the summation of the squared distance) and L_∞-norms. The problem with the L_∞-norm can be formulated as a three-variable linear programming problem and can be solved in linear time [Dy86, Me84]. The problem can also be solved optimally in any fixed dimension. The well known *least-squares* method finds the solution for the L_2-norm in linear time. Finally, Imai, Kato and Yamamoto [IKY89] recently presented an optimal linear time algorithm to solve the problem under the L_1-norm.

The problem is more complicated with the Euclidean *orthogonal* distance. Lee and Wu [LW86] proved a $\Omega(n \log n)$ time lower bound for the L_∞-norm. Few optimal algorithms have been problem. Hence, we can use the linear time algorithm for linear programming of Dyer and Megiddo [Dy86, Me84] to solve the problem. Recently Seidel [Se90] presented a much simpler and more efficient randomized version of this algorithm which runs in linear expected time. It is interesting to note that the problem can also be stated in higher dimensions where the customers become hyperplanes and can still be solved in optimal linear time for fixed dimensions. Finally we mention that under the Euclidean orthogonal distance this problem has an interesting geometrical interpretation: find the smallest disc that intersects each of a given set of lines. Similarly, under the Euclidean vertical distance we require the shortest vertical line segment that intersects every line.

For the L_1-norm, the location problem cannot be formulated as a linear programming problem in a fixed dimension. Nevertheless, it is possible to find a solution in $O(n^2)$ time for both distance notions. The problem can be stated as follows:

$$\text{minimize} \quad \Sigma\, w_i \left| a_i x + b_i y + c_i \right|$$

or,

$$\text{minimize} \quad \Sigma\, e_i$$

subject to

$$w_i\,(a_i x + b_i y + c_i) \ \le \ e_i$$

$$w_i\,(a_i x + b_i y + c_i) \ \ge \ -e_i.$$

From this formulation, it is easy to prove that an optimal solution for the facility location must be one of the intersections of the line-customers [MN80]. So, there is an obvious brute force algorithm which solves the problem in $O(n^3)$ time. However, using the *topological-sweep* algorithm of Edelsbrunner and Guibas [EG89] it is possible to enumarate all candidate solutions and evaluate the objective function in $O(n^2)$ overall time and $O(n)$ space [HR88].

3.3 Simple convex objects as customers

Simple-object-customers such as line segments and small convex polygons are more difficult to deal with. There is no obvious way to express the facility location problem as a linear programming problem. Therefore, if we want to find a solution, we have to rely on some geometrical properties of the problem. For this reason we will look only at the unweighted Euclidean 1-center problem. In this case, the L_∞-norm is used and the distance between a point X and a customer s, denoted $d_E(X, s)$, is given by $\inf_{p \in s} d_E(X, p)$. The solution presented here uses one of the most fundamental structrures in computational geometry known as the *furthest point Voronoi diagram* and it is an extension of the algorithm of Shamos and Hoey [SH75] for finding the smallest circle enclosing a set of points. The furthest point Voronoi diagram for a set S of n sites, denoted $FPVD(S)$, is a subdivision of the plane into unbounded regoins $V(s_i)$ such that $V(s_i)$ is the locus of points further from the site s_i than any other site in S. Leven and Sharir [LV87] gave long lists of properties of the $FPVD(S)$ under the above definition of distance. For our algorithm, we need important however is the case of locating “undesirable” or obnoxious facilities. In this case instead of *minimizing* the *maximum* distance between the facility and its customers as we have done so far, we would like to *maximize* the *minimum* distance. This would be true for example with facilities such as smelly garbage dumps, dangerous chemical factories and nuclear power plants. This *maximin* facility location problem also has a simple geometrical interpretation: it is the largest empty circle, i.e., the facility location is the center of the largest circle that does not contain any customers in its interior. As stated this problem is of course trivial since we just have to place the facility at infinity. Therefore to make the problem meaningful and useful in practice we add a constraint that the facility should be located in some specified polygonal region perhaps corresponding to the boundary of a country or some arid land inside a country. Toussaint [To83] gives an $O(n^2 \log n)$ time algorithm for locating such a facility for n point customers where the facility is constrained to lie in a specified simple polygon of n vertices. Bhattacharya and Elgindy [BE86] reduced this complexity to $O(n \log n)$. If the constraint polygon is convex the problem can be solved in $O(n \log n)$ time [PS85]. It may also be the case that the “customer” is a simple polygon as for example the boundary of a lot in which case the obnoxious facility location problem reduces to finding the largest circle that can be inscribed in a simple polygon. Karzakis and Karagiorgis [KK85] give an $O(n^2)$ time algorithm for solving this problem. It should be pointed out however that we can do better. The largest inscribed circle must have its center at one of $O(n)$ bifurcation points in the *medial axis* of P and therefore can be obtained in $O(n)$ time once the *medial axis* of P is found. Furthermore, the *medial axis* of P can be computed in $O(n \log n)$ time [Le82]. Finally we mention that if P is convex finding the largest inscribed circle in P can be formulated as a linear program in three variables and can thus be solved in $O(n)$ time [Me84].

3.2 Lines as customers

Let $C = \{l_1,..., l_n\}$ be a family of n non-vertical lines where each l_i is defined by the equation $a_i x + b_i y + c_i = 0$ such that $a_i^2 + b_i^2 = 1$. Two notions of distance between the point-facility and the line-customer come to mind easily: the Euclidean orthogonal distance and the Euclidean vertical distance. Although they are different it is possible to solve both problems in a similar way. Hence, we consider the point-facility location problem for line-customers for the L_1 and L_∞-norm under the Euclidean orthogonal distance.

For the L_∞-norm, the location problem can be formulated as a three-variable linear programming problem stated as follows:

$$\text{minimize} \quad \max w_i \left| a_i x + b_i y + c_i \right|$$

or, equivalently, as

$$\text{minimize} \quad z$$

$$\text{subject to}$$

$$w_i (a_i x + b_i y + c_i) \leq z$$

$$w_i (a_i x + b_i y + c_i) \geq -z.$$

Therefore, the location problem can de reduced to a three-variable linear programming

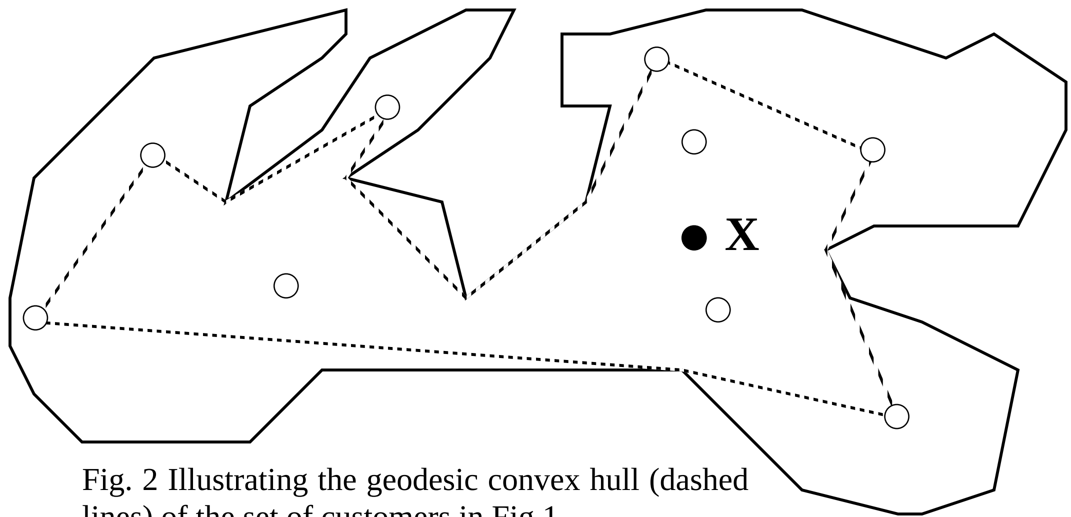


Fig. 2 Illustrating the geodesic convex hull (dashed lines) of the set of customers in Fig.1.

is easier. It is possible to solve it optimally in linear time for both the L_∞-norm [EH72] and the L_1-norm [Ba84].

The *geodesic center* of a polygon is a generalization of the Euclidean facility location problem. The geodesic center of P, denoted by $C_G(P)$, is a point in P which minimizes the maximum geodesic distance to any other point in P. Such a distance is called the *geodesic radius* of P and denoted by $R_G(P)$. More precisely, for any point x in P define the *covering radius* of P from x as:

$$C_r(P/x) = \max_{y \in P} \{d_G(x,y)\}.$$

Then the geodesic center of P is the point in P for which

$$R_G(P) = \min_{x \in P} \{C_r(P/x)\}.$$

The problem of computing the geodesic center of a simple polygon was first investigated by Asano and Toussaint [AT85] who showed that it was unique and could be computed in $O(n^4 \log n)$ time. This result was later improved to $O(n^3 \log \log n)$ time in [AT86], to $O(n \log^2 n)$ time in [PS86], and finally to $O(n \log n)$ time in [PSR89]. This formulation of the problem tacitly assumes that all points in P are an infinite family of customers, not a very realistic assumption. However, if we are given a set C of n specific point locations in the interior of P representing customers then the geodesic center of C is the geodesic center of the geodesic convex hull of C in P and can be computed in $O(n \log n)$ time [To89].

The problems discussed so far and for the most part in this paper, as well as the facility location literature in general, are concerned with what we might call “desirable” facilities in the sense that we want to *minimize* some distance function between the facility and the customers. Just as covered recently algorithms very simple to program that may turn out to be the most useful in practice. In particular there exists an algorithm which runs in time $O(n(1+t))$ where t is a non-negative integer related to the shape complexity of P and $t < n$ [To89]. The simplest of all sub-quadratic algorithms however, requiring only a page of code, runs in time $O(n(1+r))$ where r is the number of reflex vertices of P [KET90].

2.4 Geodesic Convex Hulls

In this sub-section we introduce the notion of the *geodesic* (also known in the literature as *relative) convex hull* of a set of points (customers) lying in a simple polygon P. As we shall see in the next section the geodesic convex hull is very useful in locating point-facilities using geodesic distance criteria.

Let Q be a subset of P. Q is called *geodesically convex* provided that for every pair of points $x,y \in Q$, the geodesic path between x and y constrained to lie in P also lies in Q, i.e., $GP(x,y/P) = GP(x,y/Q)$.

Let S be a set of customer-sites in P. The *geodesic convex hull* of S in P denoted by $CH_G(S/P)$, is the intersection of all *geodesically-convex* sets containing S. Refer to Fig. 2 for an illustration. Alternately we may view the *geodesic convex hull* as the *minimum-perimeter polygonal circuit* that contains S and is constrained to lie in P.

3. Point-Facility Location

A general formulation of the problem considered in this section may be stated as follows: let $C = \{c_1,..., c_n\}$ be a family of *customers*, X be the *point-facility* and f be the *cost function* given by $f(X) = g(w_1 d(X, c_1),..., w_n d(X, c_n))$ where $d(X, c_i)$ denotes the distance between the customer c_i and the point-facility X, w_i is the weight associated with customer c_i and g is a norm function combining all the weighted distances (usually the *summation* or the *maximum* function). The point-facility problem is to determine an optimal solution, say X^*, that minimizes the cost function $f(X)$. Many aspects of this problem have been investigated in operations research with point-customers and they are summarized in the next subsection. Besides theses solutions, few attempts have been made to extend this problem to more complex models of customers. To remedy this situation, we will present some extensions where the customers are *lines, line segments* or *simple objects*.

3.1 Points as customers

The point-facility location problem for point-customers with the Euclidean distance, defined as $d_E(p, q) =((x_p - x_q)^2 + (y_p - y_q)^2)^{1/2}$and the L_∞-norm (i.e. the maximum function) is known in the literature as the *Euclidean 1-center problem*. This problem has a very long history and was posed originally in 1857 by Sylvester [Sy1857]. As mentioned in the introduction, Megiddo [Me83] presented an optimal linear time algorithm to solve the unweighted case. His solution based on the *prune-and-search* technique has been extended in [Dy86, Me84] to solve optimally the weighted problem in any fixed dimension. With the L_1-norm (i.e., the summation function) this corresponds to the well known *Fermat-Weber* problem for which only heuristic solutions are known [Ba84]. For the Manhattan distance, defined as $d_M(p, q) = |x_p - x_q| + |y_p - y_q|$, the problem

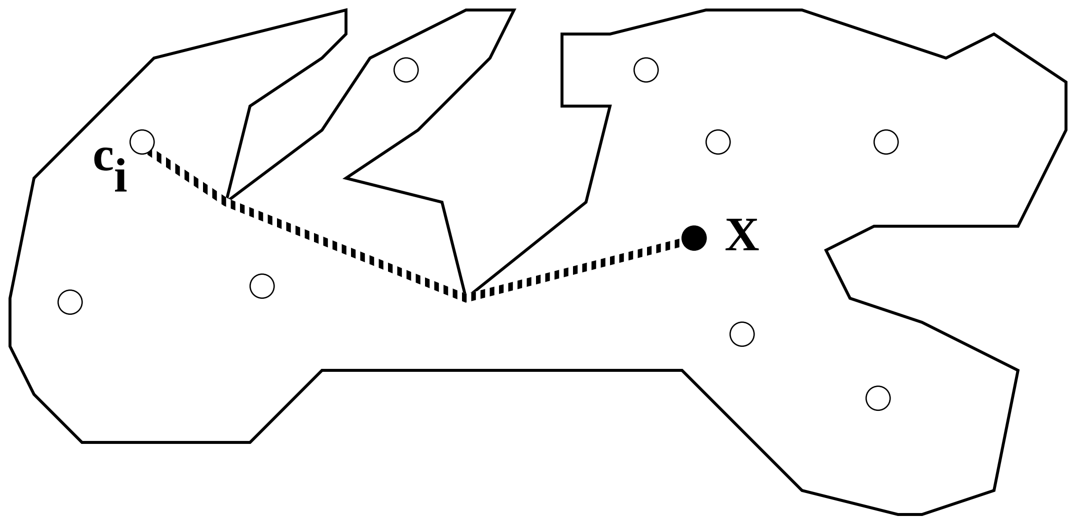


Fig. 1 Illustrating the geodesic path between a customer and a facility *X* in a bounded region such as an island.

tices of *P*. A simple polygon has a well defined interior and exterior. We will follow the convention of including the interior of a polygon when referring to *P*. The vertices of *P are* either *convex* or *reflex*. A vertex is convex if its internal angle is less than 180° and reflex if it is greater than 180°.

A *polygonal path* is a simple path consisting of a sequence of line segments. If *p* is a polygonal path, then the *length* of *p* is the sum of the Euclidean lengths of all the line segments comprising *p*. Given two points *x* and *x'* in *P* the geodesic path between *x* and *x'* denoted by $GP(x,x'/P)$ is the minimum-length polygonal path ($x=x_1,x_2,...,x_k=x'$). The length of the geodesic path is called the *geodesic distance* and is denoted by $d_G(x,x')$. Two fundamental properties of the geodesic path $GP(x,x'/P)$ are that the path is unique and its vertices x_i, i=2,3,...,k-1 are a subset of the reflex vertices of *P* [Ch82], [LP84]. Chazelle [Ch82] and Lee and Preparata [LP84] independently obtained an elegant and simple algorithm for computing $GP(x,x'/P)$ and $d_G(x,x')$ optimally in linear time provided that *P* has already been triangulated. Since polygon triangulation plays a central role in many of these geodesic facility location algorithms we devote a short sub-section to that topic next.

2.3 Triangulating Polygons

A *triangulation* of a simple polygon *P* containing *n* vertices is the decomposition of *P* with a set of *n*-3 *internal diagonals* (line segments [*x,y*] connecting pairs of vertices *x,y* of *P* such that *int*[*x,y*] lies in *int*(*P*)) into *n*-2 triangles such that the diagonals intersect each other only at their endpoints. Much work has been done to find efficient algorithms for triangulating simple polygons and the interested reader is referred to [PS85] for an introduction to the subject and to [Ch90] for a review of the most recent work. Here we mention only the most important theoretical and practical recent results. In the past fifteen years the theoretical complexity of triangulating polygons has steadily decreased from $O(n^3)$ to $O(n^2)$ to $O(n \log n)$ to $O(n \log \log n)$ and finally to $O(n)$ [Ch90]. The fastest theoretical algorithms [Ch90] however appear to be very difficult to program and it is not clear just how useful they will be in practice. On the other hand there have been discomputing such structures can be found.

2. Some Basic Computational Geometric Tools

2.1 Convex Hulls

One of the most useful structures in computational geometry is the *convex hull* of a set of points C, i.e., the smallest convex set containing C [To85a]. In designing their facility location algorithm Bass and Schubert [BS67] used the fact that the smallest enclosing circle of a set of points C is determined only by points which are extreme on the convex hull of C. This fact together with an $O(n \log n)$ time algorithm which they propose for computing the convex hull of C, and a brute force approach on the resulting extreme points leads to their complexity of $O(h^4 + n \log n)$.

In practice we are often concerned with the average or expected time complexity of an algorithm. In 1978 Akl and Toussaint [AT78] proposed a *sieve* method they dubbed *the throw-away principle* for computing convex hulls in linear expected time. The idea is to first throw away a "large" subset of the points with a very fast and simple $O(n)$ time procedure and subsequently use any standard convex hull algorithm on the remaining set. This is accomplished by choosing an appropriate fixed number (say four or eight) of directions, searching for the extreme points in these directions, constructing the convex polygon determined by connecting the extreme points in a clockwise order, and finally discarding from further consideration the points of S that fall in the interior of the resulting convex polygon. Akl and Toussaint [AT78] showed that for n points uniformly distributed in the unit square an $O(n \log n)$ *worst-case* time algorithm could be made to run in $O(n)$ *expected* time. Devroye and Toussaint [DT81] extended these results showing that for n points randomly distributed in *any* non-degenerate rectangle R in the plane, according to *any* density function f whatsoever, as long as f is zero outside R and bounded away from zero and infinity inside R, applying such a throw-away step to any algorithm will result in an overall expected complexity of $O(n)$, even if the convex hull algorithm used after the throw-away step has a worst-case complexity of $O(n^2)$. The FORTRAN code for a variant of this algorithm published by Bhattacharya and Toussaint [BT83] appears to be the fastest in practice while using only $5n$ storage space.

2.2 Geodesic Paths

In the classical facility location problem discussed in the introduction it is tacitly assumed that the space "between" the facility and the customers is not obstructed by obstacles of any sort with the result that the shortest path between a customer c_i and the point-facility X is the Euclidean distance between c_i and X. This is certainly a realistic situation if, for example, transportation is to be carried out by helicopter or airplane. For ground transportation on the other hand it is often unrealistic as boundaries of countries may have to be respected or the region of interest may be an island. In such situations it is more useful to model the customers as points in a polygonal region P and to measure the distance between customer c_i and the point-facility X by the *geodesic* distance between c_i and X, i.e., the length of the *geodesic-path* in P between c_i and X. Figure 1 illustrates the geodesic path between a facility X and a customer c_i.

A polygon P is called a *simple* polygon provided that no point of the plane belongs to more than two edges of P and the only points of the plane that belong to precisely two edges are the verefficient (in the worst case) algorithm that runs in $O(n^2)$ time.

About four years later the computer science discipline of *computational geometry* was born with the work of Michael Shamos [Sh78]. Now a fifteen-year old explosive discipline it continues to flourish at an exponentially increasing rate and make its presence felt in several areas not the least of which is facility location. Several books have already appeared on the subject. An introductory text by Preparata and Shamos [PS85] covers most of the early work. Mehlhorn [Me84] contains a subset of the material found in Preparata and Shamos along with a few different results. The combinatorial aspects of discrete and computational geometry as well as the prune-and-search methods useful in obtaining the fastest algorithms for many facility location problems are treated in depth in the book by Edelsbrunner [Ed88]. The question of visibility, of great interest to locating facilities which consist of surveillance or illumination equipment, is notoriously absent from the three texts mentioned above. However visibility is given a clear, excellent, and comprehensive treatment in the recent book by O'Rourke [O'R87]. One of the most fundamental structures in computational geometry is the Voronoi diagram and since the "birth" of computational geometry a score of variants on this structure have appeared. The book by Rolf Klein [Kl89] is entirely devoted to this subject. There have also appeared five books which are collections of papers covering almost all aspects of computational geometry. The book edited by Preparata [Pr83] contains twelve papers on early material. More recent results can be found in the two books edited by Toussaint [To85b], [To88a] and in the robotics-oriented collections edited by Schwartz et al., [SSH87] and Schwartz and Yap [SY87]. Journals are also starting to devote special issues to computational geometry such as *The Visual Computer* [To88], *Pattern Recognition Letters* [To91], and *The Proceedings of the IEEE* [To92].

The smallest enclosing circle (also minimal spanning circle) problem mentioned above as well as many other facility location problems has benefited substantially from the developments in computational geometry. Shamos [Sh78], Shamos and Hoey [SH75] and Preparata [Pr77] were the first to discover $O(n \log n)$ time algorithms, a considerable improvement over the $O(n^2)$ solution of Elzinga and Hearn [EH72]. The algorithms in [Sh78] and [SH75] have a step in which they compute the diameter of the set with an invalid diameter-algorithm and a counter-example to this diameter algorithm is given by Bhattacharya and Toussaint [BT85]. In spite of this default it is shown in [BT85] that the minimal spanning circle algorithm in [Sh78] always yields the correct solution and two alternate $O(n \log n)$ time algorithms are also given there. Lee [Le80] proposed a similar $O(n \log n)$ time algorithm for computing the minimal spanning circle. Finally, Megiddo [Me83] found an optimal $O(n)$ time algorithm for solving this problem.

In this paper we briefly survey the most recent results on facility location concentrating on versions of the problem that are probably unfamiliar to the transportation and management science community. Such versions include the standard models of *points* as customers and facilities but with *geodesic* rather than the traditional Minkowski metrics as measures of distance, as well as more complicated models of customers and facilities such as *lines*, *line segments*, *circles* and *polygonal* objects. We also consider multiple facility location problems concerned with *visibility* such as the installation of surveillance or lighting equipment. Since efficient algorithms for solving many of these problems depend heavily on some basic computational geometric structures we also provide in the next section some pointers to the literature where the most efficient algorithms for

Computational Geometry and Facility Location

Jean-Marc Robert

Godfried Toussaint

School of Computer Science
McGill University, Montreal

In this paper we briefly survey the most recent results in the area of facility location, concentrating on versions of the problem that are likely to be unfamiliar to the transportation and management science community and we explore the interaction between facility location problems and the field of computational geometry. Such versions of the problem include the standard models of *points* as customers and facilities but with *geodesic* rather than the traditional Minkowski metrics as measures of distance, as well as more complicated models of customers and facilities such as *lines*, *line segments*, *circles* and *polygonal* objects. We also consider multiple facility location problems concerned with *visibility* such as the installation of surveillance or illumination equipment. We include several new results including an optimal O(n log n) time algorithm for finding the thinnest strip that intersects a given set of polygonal customers.

1. Introduction

In the classical facility location problem [FW74] we are given a set of n points C in the plane representing customers, plants to be serviced, schools, markets, distribution sites or what have you, depending on the context in which the problem is embedded, and it is desired to determine the location X (find another point in the plane) where a facility (service, transmitter, dispatcher, etc.) should be located so as to minimize the Euclidean distance from X to its furthest customer (point in C.) Such a *minimax* criterion is particularly useful in locating emergency facilities, such as police stations, fire-fighting stations and hospitals where it is desired to minimize the worst-case response time. This problem has an elegant and succinct geometrical interpretation: find the smallest circle that encloses a given set of n points. The center of this circle is precisely the location of X. This geometric setting together with the fact that the smallest enclosing circle of C is determined by either a pair or a triplet of points in C immediately suggests a naive, brute-force, method for obtaining a solution: (1) for every pair of points determine its diametral circle, (2) for every three points determine the circle they uniquely define, (3) for every circle thus formed determine if no other points lie outside it and (4) out of all such “full” circles select the smallest encountered. This algorithm has a time complexity of $O(n^4)$. An improved adaptive algorithm for this problem was proposed by Bass and Schubert in 1967 [BS67] and although no complexity analysis is given it can be shown[To85a] that their algorithm runs in time $O(h^4 + n \log n)$, where h is the number of points that are extreme points of the *convex hull* of C. In 1972 Elzinga and Hearn [EH72] proposed a more